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ON CONSERVATIVE MINIMAL OPERATIONS

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1. INTRODUCTION

In this paper we use the terminology and notations of [6] together with the common universal algebraic language [3]. Thus, an operation f is *minimal* if the clone $[f]$ generated by f is minimal, and f is of minimal arity among the nontrivial operations in $[f]$. After QUACKENBUSH [5], we call an operation g on a set A *conservative* if every subset of A is closed under g.

In [6] Rosenberg provided a classification of minimal operation on finite sets. Namely, such an operations is always of one of the following five types:

1) unary;
2) binary idempotent;
3) ternary majority;
4) semiprojection;
5) $x+y+z$ in a boolean group.

No nontrivial operations of type 1) or 5) are conservative on at least three element sets. Hence for the study of conservative minimal operations we have to consider the cases 2), 3), and 4) only. Here we settle the cases 2) and 3), i.e. we determine all conservative minimal binary

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operations and all conservative minimal ternary majority operations on finite sets.

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2. PREPARATORY REMARKS

The operations we consider are defined on a finite set $\underline{n} = \{0,1,\ldots,n-1\}$ $(n > 1)$. The sign $P_k(\underline{n})$ stands for the set of all k-element subsets of $\underline{n}$. For ternary majority operations (shortly: *majority operations*) on $\underline{3}$ we use the notation of [1]; i.e. m_i denotes the majority operation f with $3^5\cdot f(0,1,2)+3^4\cdot f(0,2,1)+3^3\cdot f(1,0,2)+$ $+3^2\cdot f(1,2,0)+3\cdot f(2,0,1)+f(2,1,0) = i$. The *range* of a majority operation f is $\{f(i,j,k): i \neq j \neq k \neq i\}$.

Attributes of an operation f on $\underline{n}$ will be extended to the algebra $\langle\underline{n};f\rangle$; thus, this algebra will be termed minimal, essentially k-ary, etc. if f is minimal, essentially k-ary, etc. We shall formulate our results in terms of algebras.

Let $A \subseteq \underline{n}$; we write f_A for the restriction of f to A. Clearly, a k-ary conservative algebra $\langle\underline{n};f\rangle$ is determined uniquely by the set of its k-element subalgebras $\langle A;f_A\rangle (A\in P_k(\underline{n}))$.

For fixed k, consider a full set R_k of representatives of isomorphism classes of all k-ary algebras $\langle\underline{k};g\rangle$. It will be appropriate for our aims to include

$$\langle\underline{2};e_1^2\rangle,\ \langle\underline{2};e_2^2\rangle,\ \langle\underline{2};\vee\rangle$$

in R_2, and

$$\langle\underline{3};m_i\rangle,\ i = 0,44,424,624,109,255,\\ 325,39,253,327,111,37$$

in R_3. Denote by M the set of these twelve majority operations.

For a k-ary conservative operation f on $\underline{n}$ and an arbitrary $A\in P_k(\underline{n})$ there exists a unique k-ary algebra $\langle\underline{k};g\rangle$ in R_k such that $\langle A;f_A\rangle \simeq \langle\underline{k};g\rangle$. The set of these algebras $\{\langle\underline{k};g\rangle: A\in P_k(\underline{n})\}$ is called *the spectrum of* $\langle\underline{n};f\rangle$; we denote it by Spec$\langle\underline{n};f\rangle$. We shall characterize conservative minimal algebras by their spectra.

Let f_1 and f_2 be k-ary resp. l-ary operations on $\underline{n}$ such that $f_1\in[f_2]$. Then there exists a k-ary term (= polynomial symbol) t of type $\langle l\rangle$ such that f_1 is the result of substituting f_2 for the l-ary operation symbol in t, in sign: $f_1 = t(f_2)$. In this case we say that we *apply* t *to* f_2. The result of succesive application of t_1 then t_2 to f is denoted by $t_2t_1(f)$; we also write $t^i(f)$ when t is applied to f i times. For f conservative and $A \subseteq \underline{n}$ always $(t(f))_A = t(f_A)$.

We shall make use of the following terms of type $\langle 3\rangle$ (here we omit the sign of the ternary operation symbol):

$$\begin{aligned} p &= ((xyz)(yzx)(zxy)),\\ q &= ((xyz)zy),\\ r &= (z(xyz)x),\\ s &= (x(yzx)y),\\ u &= (y(zyx)x). \end{aligned}$$

Next we formulate several observations to be applied in the sequel.

(1) *Each subalgebra of a minimal algebra is either minimal or trivial* (see [6], 3.3).

(2) *A two-element conservative minimal binary algebra is isomorphic to* $<\underline{2};\vee>$.

(3) *The set of all three-element minimal majority algebras is* $\{<\underline{3};m_i>: m_i \in M\}$ *up to isomorphism.*

Denote the set $\{m_0\}$ by M_0, $\{m_i: i = 44,424,624\}$ by M_{44}, and $\{m_i: i = 109,255,325,39,253,327,111,37\}$ by M_{109}. We proved in [1] that the minimal clones generated by majority operations on $\underline{3}$ are $[m_0]$, $[m_{44}]$, and $[m_{109}]$ up to a permutation of $\underline{3}$. This combined with the following proposition gives (3).

If o is a majority operation on $\underline{3}$ then

(a) $o \in [m_0]$ iff $o \in M_0$,

(b) $o \in [m_{44}]$ iff $o \in M_{44}$,

(c) $o \in [m_{109}]$ iff $o \in M_{109}$.

(a) The range of m_0 consists of 0 only, hence by Lemma 2 in [1] the same holds for each ternary operation in $[m_0]$; thus the set of nontrivial ternary operations in $[m_0]$ is $\{m_0\} = M_0$.

(b) $M_{44} \subseteq [m_{44}]$ since $m_{424}(x,y,z) = m_{44}(y,x,z)$ (shortly: $m_{424} = (m_{44})_{(01)}$), and $m_{624} = (m_{44})_{(02)}$; further, m_{44} is a homogeneous operation, hence all members of $[m_{44}]$ are homogeneous; however, the set of homogeneous majority operations on $\underline{3}$ is exactly M_{44} (see [2]).

(c) $M_{109} \subseteq [m_{109}]$ since $m_{255} = (m_{109})_{(01)}$, $m_{325} = q(m_{109})$, $m_{39} = (m_{325})_{(01)}$, $m_{253} = (m_{39})_{(012)}$, $m_{327} = (m_{39})_{(021)}$, $m_{111} = (m_{39})_{(02)}$, $m_{37} = (m_{39})_{(12)}$. On the other hand, the range of m_{109} is $\{0,1\}$ and, as m_{109} is minimal, each ternary operation in $[m_{109}]$ has this property; also the operations in $[m_{109}]$ share the property of m_{109} to be invariant under the transposition (01) of the base set $\underline{3}$. Comparing this with the list of majority functions on $\underline{3}$ with range $\{0,1\}$ ([1], Table 3) we conclude that the set of nontrivial ternary operations in $[m_{109}]$ is M_{109}, as required.

(4) *A conservative algebra* $<\underline{n};f>$ *is not minimal if there exists a nontrivial operation* g *on* $\underline{n}$ *such that*

1) $g \in [f]$;

2) *there are subsets* $A,B \subseteq \underline{n}$ *with* $<A;f_A> \not\cong <B;f_B>$ *and* $<A;g_A> \cong <B;g_B>$.

Indeed, $f \notin [g]$; in the contrary case, an isomorphism between $<A;g_A>$ and $<B;g_B>$ would also be an isomorphism between $<A;f_A>$ and $<B;f_B>$.

A k-ary operation f is called *sharp* if it is essentially k-ary, and every $g \in [f]$ with arity $<k$ is trivial. (Thus, minimal operations are the same as sharp operations generating minimal clones.)

(5) *A conservative algebra* $<\underline{n};f>$ *with* f *k-ary is not minimal if there exists an* l*-ary* $(l > k)$ g *in* $[f]$ *which is sharp on some* $A \subseteq \underline{n}$, $|A| > 1$.

Indeed, f_A is not trivial and $f_A \notin [g_A]$ whence $f \notin [g]$.

The following fact is obvious:

(6) *An essentially k-ary algebra* $\langle \underline{n}; f \rangle$ *is minimal if and only if*

1) *for each nontrivial* $g \in [f]$ *there exists an essentially k-ary* g' *in* $[g]$, *and*

2) *if* $g \in [f]$ *and* g *is essentially k-ary then* $f \in [g]$.

An at least ternary operation f on $\underline{n}$ is a *near-unanimity operation* [4] if $f(x,y,\dots,y) = f(y,x,y,\dots,y) = \dots = f(y,\dots,y,x) = y$ identically holds in $\langle \underline{n}; f \rangle$.

(7) *If* m *is a majority operation then any nontrivial* f *in* $[m]$ *is a near-unanimity operation.*

Call a term t of type $\langle 3 \rangle$ *regular* if it is nontrivial and no occurrence of the operation symbol in it has two graphically equal arguments. Applying a nontrivial term t to a majority operation we always can suppose that t is regular. Thus, we can suppose $f = t(m)$ with t regular and use induction on the length (= number of occurrences of the operation symbol) of t. For t of length 1, $t(m)$ is a majority operation, hence the assertion of (7) is true. Assume it is true for regular terms of length $< k$. We have $t(m) = m(t_1(m), t_2(m), t_3(m))$, i.e. $t(m)(x_1,\dots,x_n) = m(t_1(m)(x_1,\dots,x_n),\dots$ $\dots, t_3(m)(x_1,\dots,x_n))$. If t_i is regular then $t_i(m)(x,y,\dots,y) = \dots = y$ by the inductive hypothesis.

Hence if at least two t_i are regular then $t(m)(x,y,\dots,y) = \dots = y$. If, e.g., t_1 is regular and $t_2 = x_i$, $t_3 = x_j$ then $x_i \neq x_j$ since t is regular. Therefore, $t(m)(x,y,\dots,y) = m(t_1(m)(x,y,\dots,y),\dots) = m(y,\dots) = y$, etc., as at least one of the second and third arguments of m also equals y.

3. RESULTS

THEOREM 1. *A conservative binary algebra* $\langle \underline{n}; * \rangle$ *is minimal if and only if* Spec$\langle \underline{n}; * \rangle$ *is a subset of* $\{e_1^2, e_2^2, \vee\}$ *which contains* $\vee$ *but not both* e_1^2 *and* e_2^2.

PROOF. Let Spec$\langle \underline{n}; * \rangle$ meet the condition in the theorem. We may assume $P_2(\underline{n}) = R \cup S$ $(R \neq \emptyset)$ where

$$\langle A; *_A \rangle \simeq \begin{cases} \langle \underline{2}; \vee \rangle & \text{if } A \in R, \\ \langle \underline{2}; e_1^2 \rangle & \text{if } A \in S. \end{cases}$$

We shall apply (6). Consider a nontrivial essentially l-ary g in $[*]$. As the unique essentially l-ary operation in $[\vee]$ is $x_1 \vee \dots \vee x_l$, we have $g(x_1,\dots,x_l) = x_1 * \dots * x_l$ on every $A \in R$. Then $g(x_1,x_2,\dots,x_2)$ $(\in [g]) = x_1 * x_2$ on every $A \in R$. Thus $[g]$ contains an essentially binary operation, and 1) of (6) holds for $\langle \underline{n}; * \rangle$.

Further, the essentially binary operations in $[*]$ are $*$ and its dual (i.e. $x_2 * x_1$), whence 2) of (6) holds, too. Therefore, $\langle \underline{n}; * \rangle$ is minimal.

Now let $\langle \underline{n}; * \rangle$ be minimal. (1) and (2) imply Spec$\langle \underline{n}; * \rangle \subseteq \{e_1^2, e_2^2, \vee\}$. Suppose $\vee \notin$ Spec$\langle \underline{n}; * \rangle$. As $*$ is not trivial, we have $P_2(\underline{n}) = U \cup V$ $(U, V \neq \emptyset)$ where

$$\langle A; *_A \rangle \simeq \begin{cases} \langle \underline{2}; e_1^2 \rangle \text{ if } A \in U, \\ \langle \underline{2}; e_2^2 \rangle \text{ if } A \in V. \end{cases}$$

There is a proper subset S in $\underline{n}$ which is maximal with respect to the property that $A \in U$ for each $A \in P_2(S)$. Let $b \in \underline{n} \setminus S$; then there exists an $a \in S$ with $\{a,b\} \in V$ and another $c \in S$ with $\{a,c\} \in U$. Consider the operation $h \in [*]$ defined by $h(x,y,z) = (x*y)*(z*x)$. Then $h(x,y,z) = x$ on every $A \in P_2(\underline{n})$, i.e. the at most binary operations in $[h]$ are trivial. However,

$$h(a,b,c) = \begin{cases} b \text{ if } \{b,c\} \in U, \\ c \text{ if } \{b,c\} \in V; \end{cases}$$

hence h is essentially ternary and sharp on $\{a,b,c\}$. In virtue of (5), $\langle \underline{n}; * \rangle$ is not minimal, a contradiction. Hence $V \in \text{Spec}\langle \underline{n}; * \rangle$.

Let $C \in P_2(\underline{n})$ be such that $\langle C; *_C \rangle \simeq \langle \underline{2}; V \rangle$ and suppose that $e_1^2, e_2^2 \in \text{Spec}\langle \underline{n}; * \rangle$. This means that there are $A, B \in P_2(\underline{n})$ with $\langle A; *_A \rangle \simeq \langle \underline{2}; e_1^2 \rangle$, $\langle B; *_B \rangle = \langle \underline{2}; e_2^2 \rangle$. Define $\circ \in [*]$ by $x \circ y = x*(y*x)$. Then $\circ$ is not trivial as $\circ_C = *_C$; further, $\langle A; *_A \rangle \not\simeq \langle B; *_B \rangle$ but $\langle A; \circ_A \rangle \simeq \langle B; \circ_B \rangle$ $(\simeq \langle \underline{2}; e_1^2 \rangle)$. By (4), $\langle \underline{n}; * \rangle$ is not minimal, a contradiction again, concluding the proof.

THEOREM 2. *A conservative majority algebra* $\langle \underline{n}; m \rangle$ $(n \geq 3)$ *is minimal if and only if* $\text{Spec}\langle \underline{n}; m \rangle$ *is a subset of* M *which contains at most one operation from each of* M_{44} *and* M_{109}.

PROOF. Assume that $\text{Spec}\langle n; m \rangle$ satisfies the condition of the theorem. Then $P_3(\underline{n}) = R \cup S \cup T$ and there is a triplet $\langle i,j,k \rangle$ such that

$$\text{(I)} \qquad \langle A; m_A \rangle \simeq \begin{cases} \langle \underline{3}; m_i \rangle,\ m_i \in M_0 \text{ if } A \in R, \\ \langle \underline{3}; m_j \rangle,\ m_j \in M_{44} \text{ if } A \in S, \\ \langle \underline{3}; m_k \rangle,\ m_k \in M_{109} \text{ if } A \in T. \end{cases}$$

Using (6), we prove that $\langle \underline{n}; m \rangle$ is minimal. First we show that for each nontrivial $g \in [m]$ there exists an essentially ternary operation in $[g]$. The nontrivial operations in $[g]$ are at least ternary, and, as g contains a minimal clone, there is a minimal operation f in $[g]$. By Rosenberg's classification theorem for minimal operations quoted in the introduction, if f is not a majority operation then it is either a semiprojection or $x+y+z$ in a boolean group. However, none of these possibilities can hold because f is a near-unanimity operation by (7). Hence f is a ternary (majority) operation, as needed.

Now consider an arbitrary nontrivial ternary $m' \in [m]$. Then there are $m_{i'} \in M_0, m_{j'} \in M_{44}, m_{k'} \in M_{109}$ such that

$$\text{(II)} \qquad \langle A; (m')_A \rangle \simeq \begin{cases} \langle \underline{3}; m_{i'} \rangle \text{ if } A \in R, \\ \langle \underline{3}; m_{j'} \rangle \text{ if } A \in S, \\ \langle \underline{3}; m_{k'} \rangle \text{ if } A \in T, \end{cases}$$

and, for every $A \in P_3(\underline{n})$, the isomorphism in (II) is the same as that in (I). We shall be done if we show that there is a term t of type $\langle 3\rangle$ such that $t(m_{i'}) = m_i$, $t(m_{j'}) = m_j$, $t(m_{k'}) = m_k$, because then

$$\langle A;(t(m'))_A\rangle = \langle A;t((m')_A)\rangle \cong \begin{cases} \langle \underline{3};t(m_{i'})\rangle = \langle \underline{3};m_i\rangle & \text{if } A\in R,\\ \langle \underline{3};t(m_{j'})\rangle = \langle \underline{3};m_j\rangle & \text{if } A\in S,\\ \langle \underline{3};t(m_{k'})\rangle = \langle \underline{3};m_k\rangle & \text{if } A\in T,\end{cases}$$

with the same isomorphisms as (in (II) and) in (I), and hence $t(m') = m$. Since $m_i = m_{i'} = m_0$, and $t(m_0) = m_0$ for any nontrivial ternary t, we have to take care of m_j and m_k only. We shall do this in two steps: first we find a term t_1 with

(III) $\quad t_1(m_{j'}) = m_{44},\ t_1(m_{k'}) = m_{109},$

and then a t_2 with

(IV) $\quad t_2(m_{44}) = m_j,\ t_2(m_{109}) = m_k.$

We can check that, for $m_{k'}\in M_{109}$, $prqp(m_{k'}) = m_{109}$. On the other hand, $prqp(m_{j'}) = m_{j'}$, if $m_{j'}\in M_{44}$. Further, $p^2s(m_{109}) = m_{109}$, $p^2s(m_{424}) = m_{624}$, $p^2s(m_{624}) = m_{44}$.

Define t_1 by

$$t_1 = \begin{cases} prqp & \text{if } j = 44,\\ (p^2s)^2prqp & \text{if } j = 424,\\ p^2s\ prqp & \text{if } j = 624.\end{cases}$$

In view of the above remarks about $prqp$ and p^2s, (III) is fulfilled. Finally, in order to construct t_2 with (IV) it

suffices to find a t_{21} with $t_{21}(m_{109}) = m_k$, $t_{21}(m_{44}) = m_{44}$ and another term t_{22} such that $t_{22}(m_k) = m_k$, $t_{22}(m_{44}) = m_{424}$, $t_{22}(m_{424}) = m_{624}$, $t_{22}(m_{624}) = m_{44}$, since then we can choose

$$t_2 = \begin{cases} t_{21} & \text{if } j = 44,\\ t_{22}t_{21} & \text{if } j = 424,\\ t_{22}^2t_{21} & \text{if } j = 624.\end{cases}$$

It can be checked that the terms t_{21} and t_{22} given in the table below for every possible value of k are appropriate:

k	t_{21}	t_{22}
109		p^2s
255	pqr	p^2s
325	q	$qspqrp$
39	rs	rp^2s
253	s^2	sp^2
327	qr	qrp^2s
111	r	rp^2s
37	s	sp^2

This completes the proof of the minimality of $\langle \underline{n},m\rangle$.

Conversely, the spectrum of a minimal conservative majority algebra $\langle \underline{n},m\rangle$ is a subset of M by (1) and (3). For any $m_j\in M_{44}$ $u^2(m_j) = m_{424}$ holds; hence Spec$\langle \underline{n};m\rangle$ cannot contain two distinct algebras from M_{44} by (4). Similarly, for any $m_k\in M_{109}$ $prqp(m_k) = m_{109}$ holds; thus, no distinct operations from M_{109} are in Spec$\langle \underline{n};m\rangle$. The theorem is proved.