The Solitaire Army Reinspected

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The solitaire army in new circumstances

Peg Solitaire soldiers—*pegs* for short—move on a plane square lattice. A peg P can jump over a horizontally or vertically neighboring comrade Q onto a free square, removing Q at the same time. The game starts with a configuration of pegs—a solitaire army—and the aim of the moves is usually to obtain another configuration with a prescribed property, e.g., one with a unique peg on a fixed square, or with a peg on a given remote square. For the essentials on Peg Solitaire see the definitive book *Winning Ways* ([**2**], Chapter 23), where problems of both types are treated in detail. Concerning a problem of the second kind, we have the following basic result of J. H. Conway (see [**6**], pp. 23–28; [**2**], pp. 715–717, 728; and [**1**]): *No solitaire army stationed in the southern half-plane can send a scout into the fifth row of the northern half-plane, but an army of* 20 *pegs can send a scout into the fourth row.*

For the proof, to every square s of the plane assign a value $p(s)$ as follows. Let σ be the golden section, i.e., $\sigma = (\sqrt{5} - 1)/2$, so $\sigma + \sigma^2 = 1$. Fix a square s_0 in the fifth row of the northern half-plane. For any square s, let $p(s) = \sigma^k$, where k is the *Manhattan distance* between s_0 and s (this means that s_0 and s are exactly k horizontal or vertical one-square steps apart). Define the *potential of a set of squares* as the sum of values of all squares in this set, and the *potential of an army* as the potential of the set occupied by that army. The potential of any army with a peg standing on s_0 is at least 1. On the other hand, the potential of the infinite army occupying all squares of the (southern) half-plane is exactly 1; we can compute it by observing that values in every column form geometric progressions with quotient σ. The rule of moves implies that no move can increase the potential; it follows that a finite army garrisoned in the southern half-plane cannot reach s_0. This kind of reasoning will occur several times in the sequel; we call it *the Conway argument*. The remaining part of Conway's result can be shown simply by displaying how the army of 20 should be deployed, and how the pegs should move (see below).

In fact, this means that if the front line of a solitaire army looks to the north, then it can advance four rows and no more, just four units of distance both in the Euclidean and the Manhattan sense. Armies, of course, do not always fight under such plain circumstances. Their front line may look to the southwest, for example, in which case the target may be the corner square of the first quadrant. Or, the territory to be scouted may be the "half-encircled" first quadrant; then the army has two perpendicular front lines, one facing north and one facing east. Or we may have two perpendicular fronts, one facing northeast and the other one northwest. FIGURE 1a shows an original northbound army; FIGURES 1b–d show the other possibilities just mentioned.

How far can the scouts be sent in cases (b), (c), and (d)? In what follows we answer these questions. The Conway argument provides upper estimates; we show that they are sharp in every case. We also prove a fact (stated in [**2**] without proof) concerning armies with a single "mounted man." We conclude with two problems about sending scouts into an "encircled ground" (cf. [**9**]).

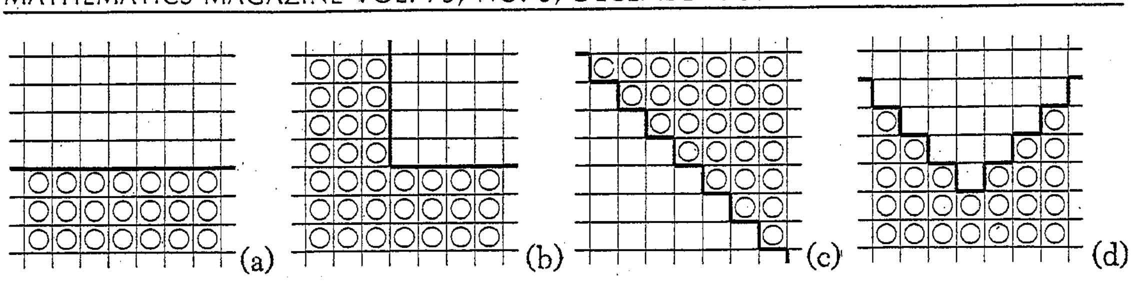

FIGURE 1
Solitaire armies in various circumstances.

Troops and advances

The aim of traditional Peg Solitaire is to evacuate a special battlefield except for one square. For this aim, the method of *packages and purges* is recommended in [2]. A *package* is a configuration of pegs which, by an appropriate sequence of moves (a *purge*), can be removed to the last peg. A simple purge is displayed in FIGURE 2a. The numbers indicate the order of moves: peg 2 jumps (onto the square 5) in the second move and returns to its starting place in the fifth move. Exponents 2, 3, etc., denote double, triple, etc., jumps.

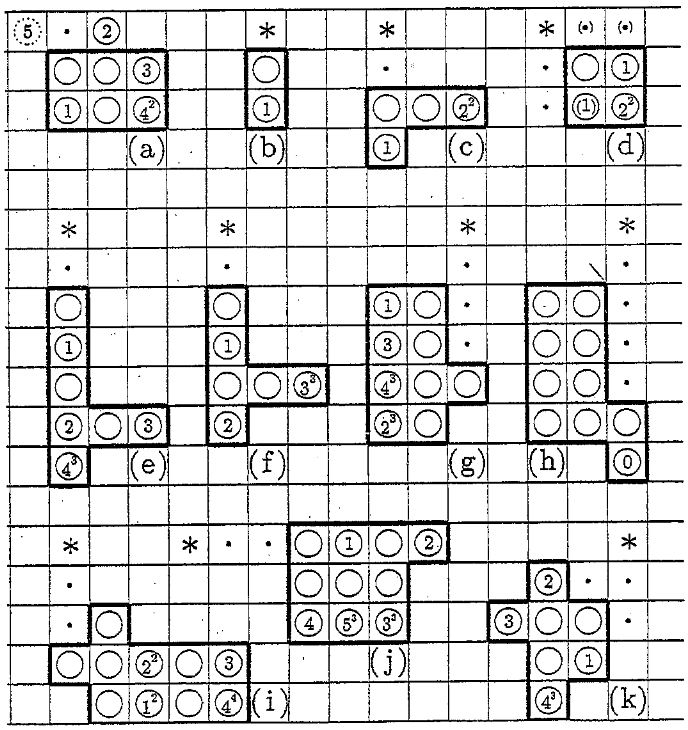

FIGURE 2
Small troops, small advances.

If the goal is to reach some remote square, instead of packages and purges we can apply the similar method of *troops and advances*. A *troop* is, in fact, a package, a member of which can reach a given distant square by a suitable sequence of moves (an *advance*). Here we list and display the troops to be deployed in order to send scouts as far as possible in cases (b), (c), and (d) of FIGURE 1. The smallest troop and advance are a pair of adjoining pegs (a *patrol*) and a single move. The patrol and other small troops are shown in FIGURE 2b, 2c, and 2d, where asterisks denote the square to be reached. We shall also operate with stronger, elite troops. They are given fitting names: *laser guns* (of length 5 and 4 in FIGURE 2e and 2f; their name comes from [5]), and *heavy guns* (of length 4 in FIGURE 2g and 2h). Guns of arbitrary length are possible and sometimes even necessary; note that the troop on FIGURE 2c is a laser gun of length 2 and a heavy gun of length 1 at the same time.

Squares marked with dots also must be free of pegs "when the guns begin to shoot." Furthermore, we call troops in FIGURE 2i, 2j, and 2k a *mustang*, a *tomahawk*, and a *halberd*, respectively.

We can compose bigger troops from smaller ones. FIGURE 3a shows that a tomahawk consists, in fact, of units 1 and 2, a laser gun, and a *block*. The squares s_1 and s_2 can be respectively occupied by the advances of unit 1 and unit 2, then a single jump of the peg on s_2 completes the operation. We denote this combined movement by the sequence $12J$. FIGURE 3b shows a *mortar* of length 6, composed from three blocks and a tomahawk. The record of its advance is $1234J^3$, where the exponent indicates that the peg on s_3 concludes the operation by a triple jump. Mortars also may be as long as needed.

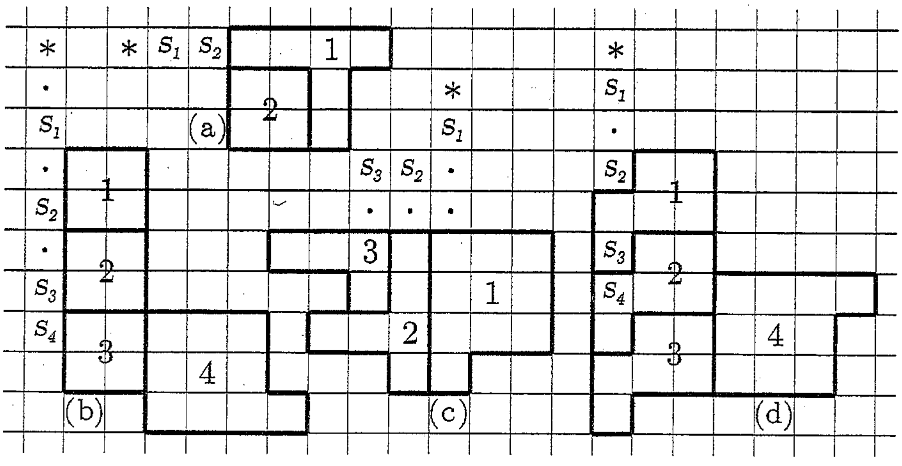

FIGURE 3
Organizing big troops from small ones.

The troop in FIGURE 3c sends a scout into the fourth row (of northern latitude), proving thereby the second part of Conway's result. Now the advance is $123J^2$. FIGURE 3d represents a *centaur*, a special troop we shall deploy soon. Its action is $1234J^3$. This notation may seem ambiguous, but the aim of the whole operation usually resolves what is left undetermined. For example, in the course of the Tomahawk Action (FIGURE 3a) block 2 moves only toward the northwest.

Pebbling and the skew front

Pebbling, a game introduced by M. Kontsevich ([3], [4], [7], [8]) is played on a square lattice in the first quadrant. Starting with one pebble in the corner square, each move consists of replacing a pebble by two, one on the north and one on the east neighboring square (FIGURE 4a). Kontsevich's problem was whether it is possible to clear the southwest triangle of 10 squares by pebbling moves (FIGURE 4b). This problem admits an "inverse" formulation as follows: Change the rule of moves so that every move undoes a possible move of ordinary Pebbling: one of the two pebbles lying as in FIGURE 4c advances south or west, and the other is removed from the field. Suppose that all squares of the first quadrant are occupied by a pebble army except for the 10 in the corner. Now the problem is whether this army is able to send a scout to the corner square. Defining the value function q by $q(s_0) = 1$ for s_0 the corner square, and $q(s) = 2^{-k}$ if the Manhattan distance between s and s_0 equals k, we can apply the Conway argument to prove that the pebble army cannot reach s_0. Indeed, the potential of the entire quadrant is 4, that of the 10-square triangle is 13/4, and hence the potential of the given pebble army is 3/4, which is less than $q(s_0)$.

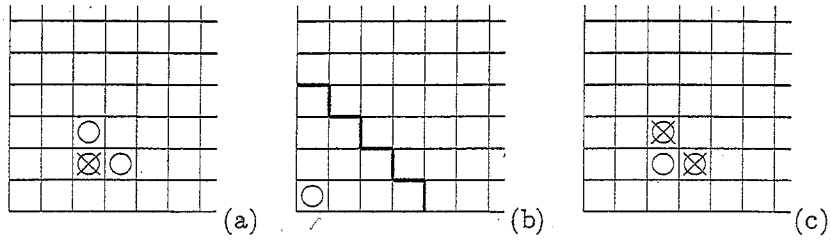

FIGURE 4
Pebbling and unpebbling.

What if, instead of pebbles, our army is recruited from tough solitaire pegs? Then, in order that the potential should not increase by (solitaire) moves, we must return to the old value function $p(s) = \sigma^k$. In this case, the Conway argument shows that an army whose skew front is at Manhattan distance 7 from s_0 cannot reach s_0. Indeed, as $\sigma^k + \sigma^{k+1} + \cdots = \sigma^{k-2}$, the potential (counted by columns) of such an army is $8\sigma^5 + \sigma^6 + \sigma^7 + \cdots = .867\cdots < 1$. If we reinforce the army with an additional (skew) line of pegs, reducing to 6 the distance between the front line and s_0, then its potential grows to $7\sigma^4 + \sigma^3 = 1.257\cdots > 1$ (FIGURE 5a).

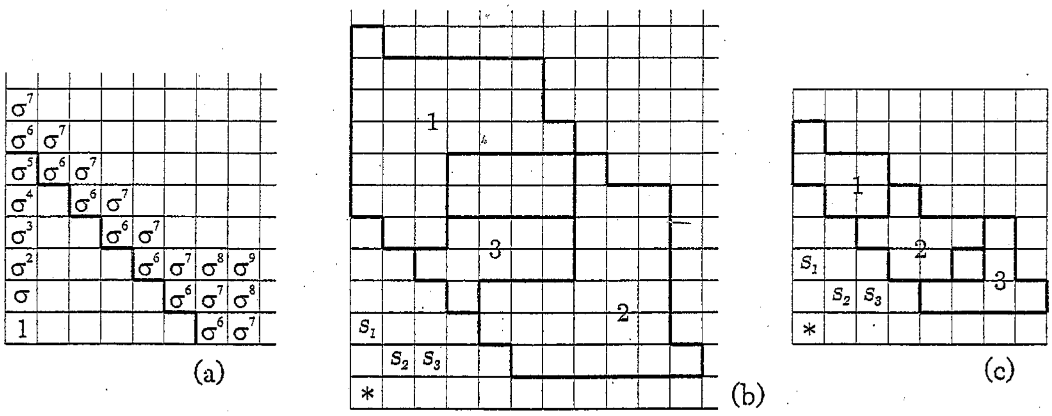

FIGURE 5
Scouting the southwestern half-plane.

Thus, sending a scout at distance 6 from the skew front-line is not prohibited by the Conway argument. It actually can be done by a joint endeavor $123JJ$ of two centaurs and a mustang; see FIGURE 5b.

We can extend the battlefield to the whole square plane without really changing the situation. If s_0 is at distance 7 from the skew front then the potential of the infinite army occupying all the squares on and behind this front equals

$$8\sigma^5 + \sigma^4 + 2(\sigma^7 + \sigma^9 + \cdots) = 8\sigma^5 + \sigma^4 + 2\sigma^6 = .978\cdots < 1.$$

This means that pegs stationed outside the first quadrant cannot provide essential help to their comrades within that quadrant.

We can also restrict the battlefield, for instance to the 8×8 chessboard. Let s_0 be the lower left corner square ("a1" in chess notation). Setting the skew front at distance 6 from s_0, the Conway argument now proves that no scout can reach s_0. However, if the distance is 5, that is, the front-line is the "a6–f1 diagonal," then s_0 is accessible for a troop of 19 pegs, namely the one consisting of a laser gun, a heavy gun, and a halberd, displayed on FIGURE 5c, by the advance $123JJ$. The number 19 cannot be decreased. We omit the simple but tedious proof we know of the latter fact, based on the Reiss theory of Peg Solitaire ([**2**], pp. 708–710).

Two fronts, horizontal and vertical

Now suppose we have the first quadrant to be reconnoitered; all other squares may be held by the solitaire army. The square at distance 5 from both fronts—square $(5,5)$ in short—can be reached by troops we introduced earlier as indicated on FIGURE 6a (cf. FIGURE 3). The advance is $1234'4''56J^5$; here $4'$ converts $4''$ into a tomahawk.

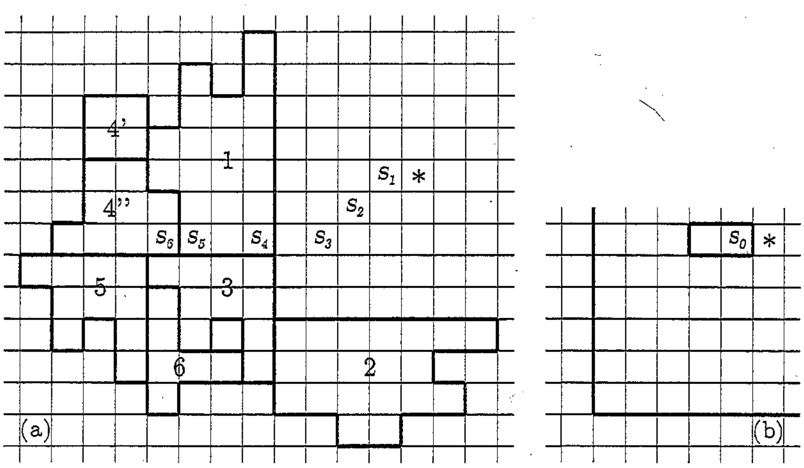

FIGURE 6
Scouting the first quadrant: possible and impossible.

Concerning square $(6,6)$, we must slightly modify the Conway argument. First let $s_0 = (6,6)$. Then the potential of our army equals $2\sigma - \sigma^8$ (the sum of potentials of the occupied half-planes minus the potential of the third quadrant) which exceeds 1, proving thus nothing. Observe, however, that in order to reach $(6,6)$, we have to send a patrol onto squares $(4,6)$ and $(5,6)$ to perform the final jump (FIGURE 6b). Letting $s_0 = (5,6)$, the potential of the patrol is $1+\sigma$, while that of the all army equals $1+\sigma-\sigma^7$, showing that $(6,6)$ is inaccessible.

What about sending a scout onto $(n,5)$, where $n > 5$? For this aim, tricks like the one applied for the case $(5,5)$ can be devised for $n = 6,7,8,9$. Fortunately, an ingenious observation in [**5**], which we call *the Eriksson–Lindström lemma*, makes them unnecessary. This lemma enables us to show: *In the case of two fronts, one horizontal and one vertical, a scout can be sent onto* $(n,5)$, *for any positive integer* n.

Partly following [**5**], we call an army, holding a finite part of some quadrant plus a single square adjacent to the border of this quadrant, a *quasi-quadrant with outpost*. Now the Eriksson–Lindström lemma says: *For any positive* n *there exists a quasi-quadrant with outpost at distance* n *from the corner square of the quasi-quadrant, which can send a scout onto the square marked by an asterisk in* FIGURE 7. The same figure also illustrates the proof for the case $n = 7$.

Here the heavy guns $1,2,\ldots,6$ complete the laser guns $7,9,11,\ldots,17$ by an additional square each. Then the echelon of all laser guns $7,8,9,\ldots,18$ produces the staircase $s_7, s_8, s_9, \ldots, s_{18}$. The heavy gun 19 sends a scout onto s_{19}, which finishes the action by a twelvefold jump. This method works for $n \geq 3$; the cases $n = 1$ and $n = 2$ are very simple.

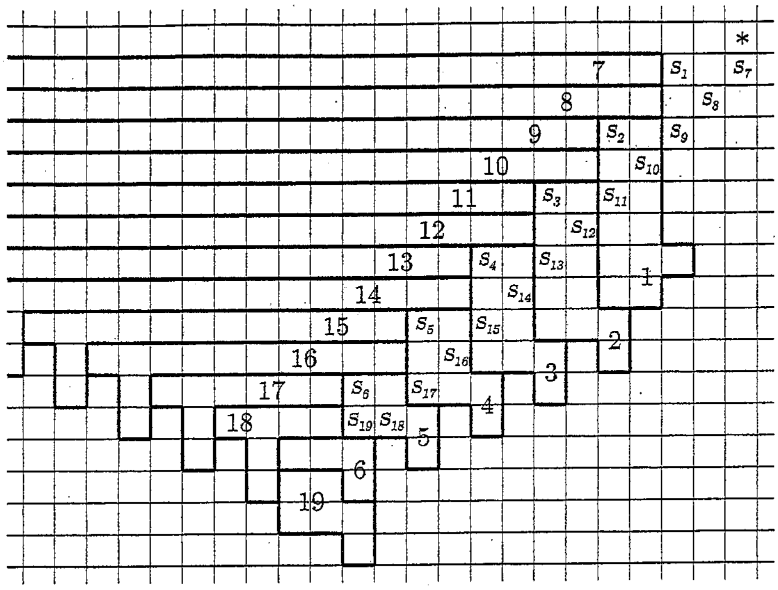

FIGURE 7
A special quasi-quadrant with outpost at distance 7 from the corner square.

Substituting "half-plane" for "quadrant" in the above definition, we get the notion of a *quasi-half-plane with outpost*. FIGURE 8 demonstrates the surprising fact that *a sufficiently large quasi-half-plane with outpost on an arbitrary square can send a scout onto any square at distance* 5 *from the border line*. Here 1 and 4 are quasi-quadrants with outposts, the Eriksson–Lindström lemma guarantees that they can send pegs to s_1, resp. s_4, and the protocol of the advance is $12J34J^2$. As a consequence, we obtain the promised result on the accessibility of $(n, 5)$ for arbitrary n.

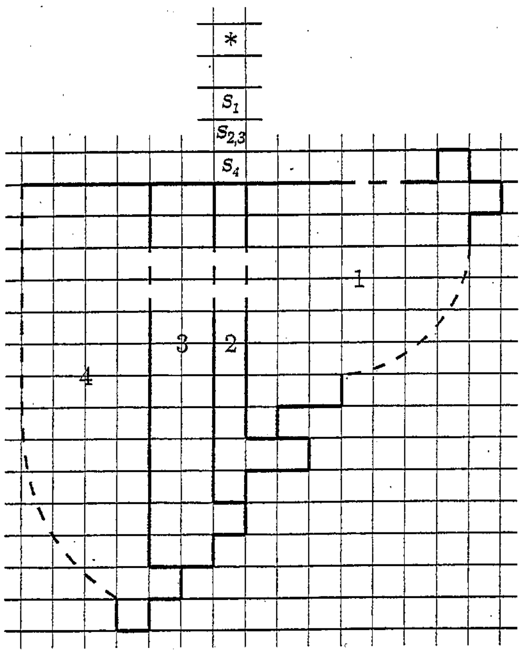

FIGURE 8
A quasi-half-plane with outpost sends a scout into an arbitrarily remote square of the northern fifth row.

Combining FIGURE 7 and 8 we see that the size of the quasi-half-plane with outpost we need for sending a scout onto the fifth row grows very fast with the distance between the outpost and the target square. An easy calculation shows that, for a distance of 67, we need an army of more than one million pegs!

A solitaire army with a solitary mounted man

In **[2]**, p. 717, one reads: "... we once showed that if any man of our army is allowed to carry a comrade on his shoulders at the start, then no matter how far away the extra man is, the problem [of sending a scout from the southern half-plane to a place in the fifth row of the northern half-plane] can now be solved." As we could not find any proof in the literature, we include one here.

Suppose there is a mounted man (i.e., two pegs stationed in the same square) in the southern half-plane. While both of them are there, this square cannot be jumped over, but the two pegs, one by one, can make legal jumps from there. Notice that a column of even length containing a mounted man one square apart from its end can be used as a laser gun: the extra man serves as the trigger. FIGURE 9a and 9b are slightly different; they show how to reach the fifth row when the distance of the mounted man from the border of the two half-planes is even or odd, respectively. The place of the mounted man is marked by a double square. The corresponding suitable advances are $12J3J45J6J^2$ and $123J45J6J^2$. Troops 4 and 6 are appropriate quasi-quadrants with outposts, sending one man each into the *first* northern square of the column, containing the mounted man.

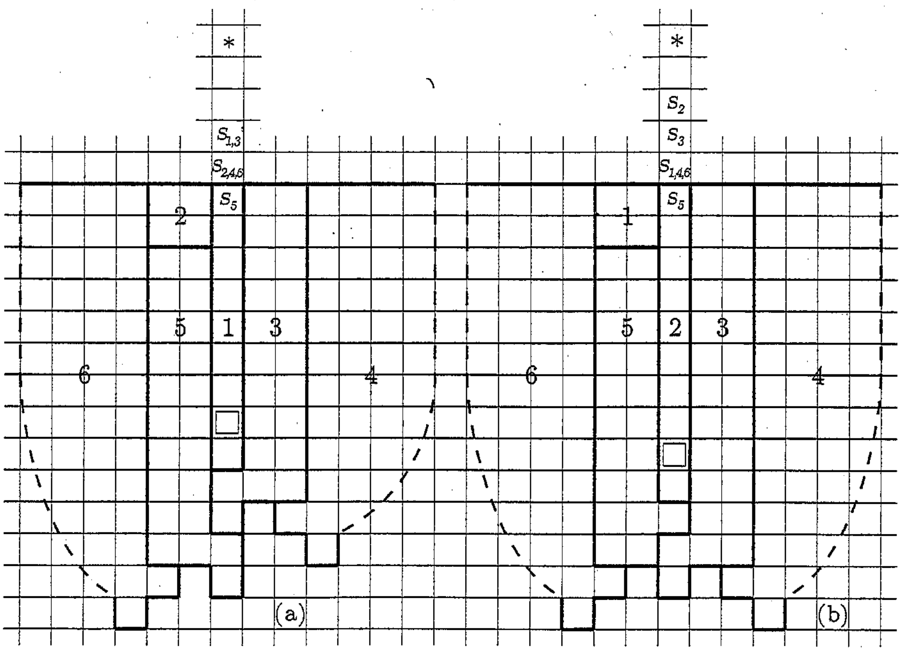

FIGURE 9
Scouting with a single mounted man.

Do not think, however, that the only possibility of reaching the fifth row is in the column of the mounted man. On the contrary: *For every pair C_1, C_2 of columns there exists a square $s_1 \in C_1$ in the southern half-plane such that a properly deployed troop in the southern half-plane with one mounted man in s_1 can send a scout onto the northern fifth square of C_2*. Instead of the lengthy full proof, we illustrate this fact in FIGURE 10 through a typical case.

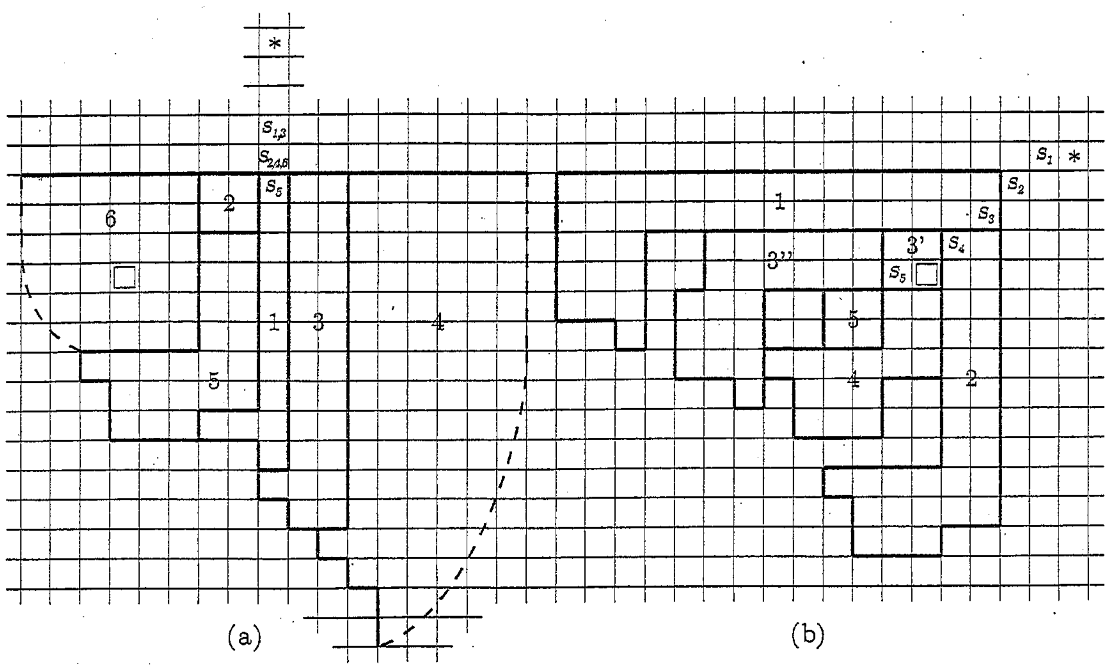

FIGURE 10
How a mounted man in a given column promotes scouting in another column.

As Figure 10a shows, it is enough to see that a suitable troop in the area 6 can send a peg onto the square s_2; then a scout reaches the target square by the advance $12J3J45J6J^2$. Figure 10b displays the suitable troop (involving several mortars) which sends a peg onto the desired square by $123'3''J45J^5$. One might worry that there is not enough space to deploy the troop in Figure 10b in the area marked by 6 in Figure 10a. However, we can guarantee the needed space in Figure 10a by replacing the guns 1, 3, and 5 by longer ones (of length 16, 18, and 12, respectively), and the quasi-quadrant with outpost (i.e., the troop 4) by a similar one whose outpost is at distance 19 from the corner.

The case of two skew fronts

Suppose that the solitaire forces are in the position of Figure 1d. Again, denote by (i, j) the square whose Manhattan distance from the left and right front lines equals i and j. Figure 11a shows that the square $(7, 7)$ marked by an asterisk is accessible: the two troops are exactly those of Figure 5b and 5c.

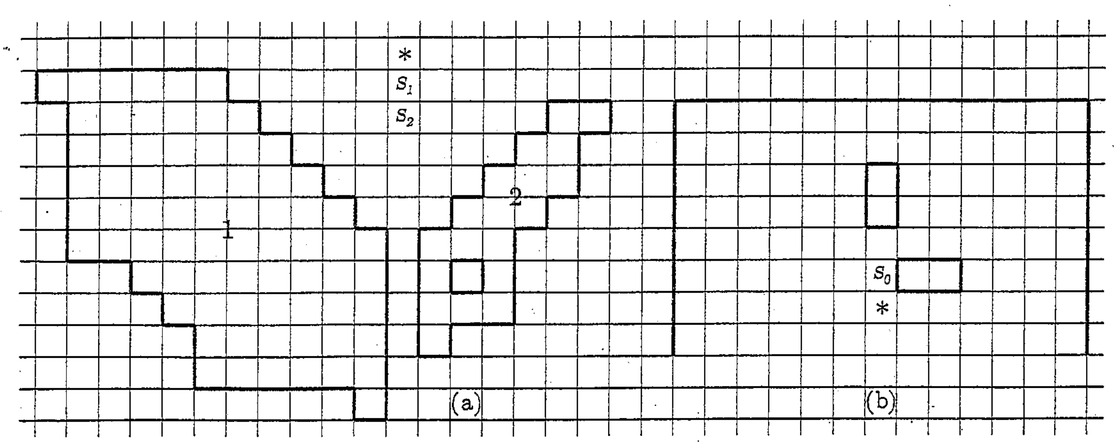

FIGURE 11
The maximal achievement in the case of two skew fronts. Patrols in encircled ground.

We prove that $(8,8)$ is inaccessible. In order to reach $(8,8)$, our army must send a patrol either to squares $(7,7)$ and $(6,6)$, or to squares $(9,7)$ and $(10,6)$. In the first case, let $s_0 = (7,7)$ and $p(s_0) = 1$; then the potential of the whole occupied area is $15\sigma^5 + 2(\sigma^7 + \sigma^9 + \dots) = 15\sigma^5 + 2\sigma^6 = 1.464\dots$, while the potential of the patrol equals $1 + \sigma = 1.618\dots$ showing that the patrol cannot be sent to the desired place. The second case is even worse: for $s_0 = (9,7)$, the potential of the patrol is the same, while that of the army will be $8\sigma^5 + \sigma^6 + 9\sigma^7 + \sigma^8 = 1.108\dots$. The squares $(n,7)$ are also inaccessible if $n > 7$: for $(9,7)$ the preceding trick works, and the original Conway argument is applicable for $n > 9$. In summary: *The squares that can be reached from two skew fronts are those at Manhattan distance no more than 6 from at least one front line, and the square $(7,7)$.*

Scouts in encircled ground

Finally, suppose that a quadrilateral area of size $n \times n$ with horizontal and vertical sides is fully encircled by a solitaire army, where n is odd. What is the maximal n such that a scout can be sent to the central square of this quadrilateral? Write $n_{\max}$ for this n. We already know that $n_{\max} \geq 9$. The Conway argument provides the upper limit $n_{\max} < 15$. Suppose $n_{\max} = 13$, and consider the last patrol, a member of which jumps into the centre. The members of this patrol are produced by two other patrols, i.e., by four pegs. They can occupy five essentially different positions; a sample is displayed on FIGURE 11b. Placing s_0 suitably, the summary potential of the army turns out to be less than that of the four pegs, contradicting the hypothesis. In our example the potential of all squares out of the 13×13 quadrilateral (i.e., of four half-planes minus four quadrants) equals

$$\sigma + \sigma^3 + 2\sigma^2 - 2\sigma^9 - 2\sigma^{11} = 1.581\dots,$$

less than the potential of the two patrols: $1 + \sigma = 1.618\dots$. Hence $9 \leq n_{\max} \leq 11$, so $n_{\max}$ is either 9 or 11. For quadrilaterals with skew sides, a similar question may be raised, and it can be treated in a similar manner. Let $n_{\max}$ be the number of squares constituting a diagonal of the maximal quadrilateral in this case. Then we obtain that $n_{\max}$ equals either 13 or 15. In both cases, the exact value of $n_{\max}$ remains unknown.

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