$x_i, x_0, \dots, x_0)$. Hence it follows

$$F_{01\dots(2n)} \cong F_{01\dots n} \times F_{0(n+1)\dots(2n)},$$

and there exists such an isomorphism under which the image of any $w_1 \in F_{01\dots n}$ [resp. $w_2 \in F_{0(n+1)\dots 2n}$] is (w_1, x_0) [resp. (x_0, w_2)].

Now copying the proof of Theorem 2 in [9] we get that $\mathcal{V}$ is equivalent to the variety of all affine modules over some ring. The unique deviation is the following. We need the fact that if $F_\omega = [M]$ is $\mathcal{V}$-free with the countable free generating set $M = \{x_0, x_1, \dots\}$, then for any finite subset M' of M the subalgebra $[M']$ is a block of some congruence in F_ω. In [9] this follows from the Hamiltonian property of $\mathcal{V}$, which is not assumed here. Observe, however, that $F_\omega \cong P_\omega$, where P_ω denotes the subalgebra of $\prod_{i<\omega} [x_0, x_i]$ consisting of the sequences which contain only a finite number of elements different from x_0. Further, there exists an isomorphism ψ such that $\psi(x_i) = (x_0, \dots, x_0, x_i, x_0, \dots)$ for $i < \omega$. Then ψ maps $F_{01\dots n}$ $(\subseteq F_\omega)$ onto

$$P_n = \{(p_1, p_2, \dots) | p_i \in [x_0, x_i];\ p_{n+1} = p_{n+2} = \dots = x_0\}.$$

Now the idempotency of $\mathcal{V}$ implies that P_n is a congruence-class in P_ω. Thus the proof is complete.

By analogous considerations, we can prove

THEOREM 3. *A variety $\mathcal{V}$ is equivalent to the variety of all affine modules over some ring if and only if $\mathcal{V}$ is idempotent and the free $\mathcal{V}$-sum with an identified element coincides with the direct product (i.e., the free $\mathcal{V}$-sum of the $\mathcal{V}$-algebras A, B with the identified element e is isomorphic to $A \times B$ and there exists an isomorphism ψ such that $\psi(a) = (a, e)$, $\psi(b) = (e, b)$, whenever $a \in A$, $b \in B$).*

Notice finally, that Theorems 2 and 3 are essentially the „affine variants" of Theorem 2.1 in [6] and Theorem 1 in [8], resp.

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VARIETIES OF MODULES AND AFFINE MODULES

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To Professor L. Rédei on his 75th birthday

Varieties of modules over semirings and rings have been characterized by JOHNSON and MANES [6] with the aid of Grätzer's notion of weak independence. Their description for modules over rings contains some conditions that guarantee the existence of subtraction directly. Here we shall show that these conditions may be replaced by some Malcev type ones. Further, a variant of the notion of independence will be used for characterizing varieties of affine modules.

1. Let $\mathcal{V}$ be a variety of algebras and $A \in \mathcal{V}$. The set $\{a_1, \dots, a_n\}$ $(\subseteq A)$ is called *weakly independent* in $\mathcal{V}$ if for any elements $b_1, \dots, b_n \in B$ $(\in \mathcal{V})$ there exists a homomorphism $\varphi: [a_1, \dots, a_n] \to B$ with $\varphi(a_i) = b_i$ for $i = 1, \dots, n$ provided there exist homomorphisms $\varphi_j: [a_j] \to B$ with $\varphi_j(a_j) = b_j$ for $j = 1, \dots, n$. In [6], the variety $\mathcal{V}$ is said to satisfy *Grätzer's condition* if

(ω) there exists a nullary operation that determines a one-element subalgebra 0 in every algebra of $\mathcal{V}$, and

(γ) for every $A \in \mathcal{V}$, $\{a_1, \dots, a_n\}$ $(\subseteq A)$ is weakly independent iff there is an isomorphism

$$f: [a_1] \times \dots \times [a_n] \to [a_1, \dots, a_n]$$

with $f(0, \dots, 0, a_i, 0, \dots, 0) = a_i$ for $i = 1, \dots, n$.

A *semiring* is an algebra $\langle R; +, \cdot \rangle$ which is a commutative monoid under addition and a monoid under multiplication such that the multiplication is distributive over the addition, and the additive unit $\mathbf{0}$ is a multiplicative zero. Let R be a semiring; a (right) *R-module* is a commutative monoid M (written additively) over which R acts according to the usual module laws, moreover, $0\varrho = m\mathbf{0} = 0$ for any $\varrho \in R$, $m \in M$, where 0 denotes the neutral element of M (see [6], [8]). For any semiring R, the class of all R-modules is a variety. An arbitrary n-ary $(n > 0)$ polynomial q of an R-module may be written in the form

$$(*) \qquad q(x_1, \dots, x_n) = x_1\varrho_1 + \dots + x_n\varrho_n,$$

where all the ϱ_i's are from R, and they are uniquely determined by q.

THEOREM 1. *A variety $\mathcal{V}$ is equivalent to the variety of all (unital right) modules over some ring if and only if $\mathcal{V}$ satisfies Grätzer's condition and anyone of the following Malcev type conditions:*

(μ_1) *the congruences of any algebra in $\mathcal{V}$ are weakly permutable (i.e., n-permutable for some $n \geqq 2$; see* E. T. SCHMIDT *[3]);*

(μ_2) *the congruence lattice of any algebra in $\mathcal{V}$ is modular;*

(μ_3) *in any algebra of $\mathcal{V}$ two congruences coincide if they have the same congruence class containing the one-element subalgebra.*

PROOF. As it has been shown by JOHNSON and MANES [6], varieties satisfying Grätzer's condition coincide up to equivalence with the varieties of modules over semirings. Further, modules over rings are the same as modules over semirings that are rings. Hence, in order to establish the necessity of our conditions it is enough to observe that varieties of modules fulfil (μ_1)—(μ_3).

On the other hand, we have to prove that the variety of all modules over the semiring R satisfies (μ_1) [resp. (μ_2), (μ_3)] only if R is a ring. Let us denote the class of all such semirings by $\mathcal{M}_1$.

If S is a semiring, the subvarieties of the variety of all S-modules are — up to equivalence — the same as varieties of modules over epimorphic images of S (see [6], [8]). Hence $\mathcal{M}_1$ is closed under epimorphic images.

Now we need the following fact whose proof is implicit in [8] (pp. 162—163): Let $\mathcal{R}$ be a class of semirings closed under epimorphic images. If every $R\in\mathcal{R}$ possesses a non-zero element having an additive inverse, then $\mathcal{R}$ is a class of rings.

In view of this fact, it remains to show that every $R\in\mathcal{M}_1$ possesses a non-zero element which has an additive inverse in R. Suppose that, on the contrary, $\alpha+\beta=\mathbf{0}$ implies $\alpha=\beta=\mathbf{0}$ for any $\alpha, \beta\in R$. It is shown by HAGEMANN and MITSCHKE [5] that the weak permutability of congruences in a variety is equivalent to the existence of ternary terms $q_1, \ldots, q_n$ for some $n\geqq 1$ so that

$$q_1(x,z,z)=x, \quad q_n(x,x,z)=z, \quad q_i(x,x,z)=q_{i+1}(x,z,z) \quad (i=1,\ldots,n-1)$$

hold identically there. Let us represent these terms in form $(*)$:

$$q_i(x,y,z)=x\varrho_{i1}+y\varrho_{i2}+z\varrho_{i3}, \quad \varrho_{ik}\in R; \quad i=1,\ldots,n; \quad k=1,2,3.$$

Considering R as an R-module and substituting $x=\mathbf{0}$, $z=1$ we get:

$$\varrho_{12}+\varrho_{13}=\mathbf{0}, \qquad \varrho_{n3}=1,$$

$$\varrho_{i3}=\varrho_{i+1,2}+\varrho_{i+1,3} \qquad (i=1,\ldots,n-1).$$

Hence $\varrho_{12}=\varrho_{13}=\mathbf{0}$, and similarly $\varrho_{23}=\ldots=\varrho_{n3}=\mathbf{0}$, a contradiction. Thus, the sufficiency of (μ_1) jointly with Grätzer's condition is proved.

The sufficiency of (μ_2) may be shown analogously using DAY's characterization of varieties for which (μ_2) holds [4]. As concerns (μ_3), we can apply FICHTNER's Malcev type theorem [7]; note, however, that the sufficiency of (μ_3) is implied also by Theorem 1 and 2 in [8].

Remark that this theorem may be considered as an alternative answer to the second part of GRÄTZER's Problem 56 in [1].

2. Let R be a semiring, M an R-module and I the set of all idempotent polynomials of M. The algebra $\langle M; I\rangle$ is called an *affine module* over R. The case when R is a ring was treated in [9]. The following remark follows from the results of [8] and [9].

Let $\mathcal{V}$ be a variety such that in any algebra of $\mathcal{V}$ every subalgebra is a block of a uniquely determined congruence. If, in addition, $\mathcal{V}$ fulfils (ω) then it is a variety of modules (of course, up to equivalence), and if in any algebra of $\mathcal{V}$ every element is a subalgebra (i.e., $\mathcal{V}$ is *idempotent*) then it is a variety of affine modules.

This observation gives the idea to replace (ω) in Grätzer's condition by idempotency with the aim of characterizing varieties of affine modules. Idempotency,

however, necessitates some modifications on (γ), as subalgebras generated by a single element are trivial in this case.

Let $\mathcal{V}$ be a variety, $A\in\mathcal{V}$, $a_0\in A$. The set $\{a_1,\ldots,a_n\}$ $(\subseteq A)$ will be called *a_0-independent* in $\mathcal{V}$, if for any elements $b_0, b_1, \ldots, b_n\in B(\in\mathcal{V})$ there exists a homomorphism $\varphi\colon[a_0,a_1,\ldots,a_n]\to B$ with $\varphi(a_i)=b_i$ for $i=0,1,\ldots,n$, provided there exist homomorphisms $\varphi_j\colon[a_0,a_j]\to B$ with $\varphi_j(a_0)=b_0$ and $\varphi_j(a_j)=b_j$ for $j=1,\ldots,n$. Let $\{a_0\}$ be a subalgebra in A; then $\{a_1,\ldots,a_n\}$ is a_0-independent if and only if $[a_0,a_1,\ldots,a_n]$ is the free $\mathcal{V}$-sum of $[a_0,a_1],\ldots,[a_0,a_n]$ with the identified element a_0 (see [2], p. 111). In the case when a_0 is a polynomial constant, a_0-independence coincides with the weak independence.

THEOREM 2. *A variety $\mathcal{V}$ is equivalent to the variety of all affine modules over some ring if and only if*

(ι) $\mathcal{V}$ is idempotent

and

(γ') for every $A\in\mathcal{V}$, $a_0\in A$, the set $\{a_1,\ldots,a_n\}$ $(\subseteq A)$ is a_0-independent iff there is an isomorphism

$$f\colon[a_0,a_1]\times\ldots\times[a_0,a_n]\to[a_0,a_1,\ldots,a_n]$$

with $f(a_0,\ldots,a_0,a_i,a_0,\ldots,a_0)=a_i$ for $i=1,\ldots,n$.

PROOF. Suppose that $\mathcal{V}$ is the variety of all affine modules over the ring R. Clearly, $\mathcal{V}$ is idempotent. Denote $[a_0,a_i]$ by Q_i $(i=1,\ldots,n)$ and let $Q_1\times\ldots\times Q_n=P$. Then, for $i=1,\ldots,n$, $P_i=\{(a_0,\ldots,a_0,p_i,a_0,\ldots,a_0)|p_i\in Q_i\}$ is a subalgebra in P and $\psi_i\colon p_i\to(a_0,\ldots,a_0,p_i,a_0,\ldots,a_0)$ maps Q_i isomorphically onto P_i. The meet $P_i\cap P_j$ $(i\neq j)$ is always $\{(a_0,\ldots,a_0)\}$. The union of all P_i's generates P; indeed, $(p_1,\ldots,p_n)=(p_1,a_0,\ldots,a_0)+\ldots+(a_0,\ldots,a_0,p_n)+(a_0,\ldots,a_0)(1-n)$ for arbitrary $p_i\in Q_i$ $(i=1,\ldots,n)$. Note that $x_1+\ldots+x_n+x_{n+1}(1-n)$ is an idempotent polynomial in any R-module, hence it is an operation in the affine R-module P. Lastly, let $\varphi_i\colon P_i\to B$ $(i=1,\ldots,n;\ B\in\mathcal{V})$ be homomorphisms with $\varphi_i(a_0,\ldots,a_0)=b_0$ and $\varphi_i(a_0,\ldots,a_0,a_i,a_0,\ldots,a_0)=b_i$ for all i. Then the mapping $\varphi\colon P\to B$ defined by

$$\varphi(a_0\varrho_1+a_1(1-\varrho_1),\ldots,a_0\varrho_n+a_n(1-\varrho_n))=$$
$$=b_0(\varrho_1+\ldots+\varrho_n-n+1)+b_1(1-\varrho_1)+\ldots+b_n(1-\varrho_n) \qquad (\varrho_1,\ldots,\varrho_n\in R)$$

is a common homomorphic extension of $\varphi_1,\ldots,\varphi_n$. Thus, P is the free $\mathcal{V}$-sum of $Q_1,\ldots,Q_n$ (up to isomorphism) with the identified element a_0.

Assume now that $a_1,\ldots,a_n$ are a_0-independent. Then also $[a_0,a_1,\ldots,a_n]$ is the free $\mathcal{V}$-sum of $Q_1,\ldots,Q_n$ with the identified element a_0, whence $P\cong[a_0,a_1,\ldots,a_n]$ and the required isomorphism f coincides with the common extension of the isomorphisms ψ_i^{-1}.

On the other hand, if the isomorphism f in (γ') exists, then $[a_0,a_1,\ldots,a_n]$ is the free $\mathcal{V}$-sum of $Q_1,\ldots,Q_n$ with the identified element a_0, whence $a_1,\ldots,a_n$ are a_0-independent.

Suppose that $\mathcal{V}$ fulfils (ι) and (γ'). Let $F_{01\ldots n}$ be the $\mathcal{V}$-free algebra freely generated by the set $\{x_0,x_1,\ldots,x_n\}$. For any natural n, $\{x_1,\ldots,x_n\}$ is x_0-independent. Hence by (γ') we have

$$F_{01\ldots n}\cong[x_0,x_1]\times\ldots\times[x_0,x_n]$$

and there exists such an isomorphism under which the image of x_i is $(x_0,\ldots,x_0,$