Stability and periodicity for differential equations with delay

Ph.D. Thesis

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Contents

Notation

$\mathbb{N}$	the set of positive numbers
$\mathbb{N}_0$	the set of nonnegative numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^n$	n-dimensional real vector space
$\mathbb{C}$	the set of complex numbers
Re	real part
Im	imaginary part
u'	the derivative of the function u
$\dot{u}$	the derivative of the function u with respect to time
$C(A, B)$	the Banach-space of continuous functions mapping from A to B, where A and B are nonempty sets
C_I	$C(I, \mathbb{R})$, where $I \subseteq \mathbb{R}$
$C^n(A, B)$	the Banach-space of n-times continuously differentiable functions mapping from A to B, where A and B are nonempty sets
$\lvert \cdot \rvert$	the euclidean norm in the n-dimensional vector space R^n
$\lVert u \rVert_I$	the maximum of $u \in C(I, \mathbb{R}^n)$, defined by $\max_{t \in I} \lvert u(t) \rvert$
u_t	the segment of the function $u \in C(I, \mathbb{R}^n)$, where $[t - r, t] \subseteq I \subseteq \mathbb{R}$, defined by $u_t(s) = u(t + s)$, $s \in [-r, 0]$
$\lVert (u, v) \rVert$	the norm of $(u, v) \in \mathcal{E} \times \mathcal{F}$, defined by $\lVert u \rVert_{\mathcal{E}} + \lVert v \rVert_{\mathcal{F}}$, where $\mathcal{E}$ and $\mathcal{F}$ are Banach spaces with norms $\lVert \cdot \rVert_{\mathcal{E}}$ and $\lVert \cdot \rVert_{\mathcal{F}}$, respectively

Chapter 1

Introduction

The thesis summarizes the results of Balázs and Krisztin [4, 3]. Article [4] is accepted for publication, an electronic version is available. Article [3] is submitted. The author has another paper [5], joint with van den Berg, Courtois, Dudás, Lessard, Vörös-Kiss, Williams and Yin, that is not presented here.

In papers [4, 3] and in the thesis we study two different types of differential equations with delay. As for the two equations different technical tools are developed, we consider them in separated chapters with slightly different notions.

The common in the two types of problems is that both are motivated by applications, and both require new, non-classical theoretical techniques. Another joint feature is that we solve open problems for both types of problems. In addition, we believe that the developed methods will turn out to be useful for a wide class of analogous models.

First we study the price model

$$\dot{x}(t) = a[x(t) - x(t-1)] - \beta|x(t)|x(t), \tag{2.1.1}$$

introduced by Erdélyi, Brunovský and Walther [9, 8, 37]. The main result is that in case $0 < a < 1$ the zero solution is globally asymptotically stable. This gives an affirmative answer for a conjecture of Erdélyi, Brunovský and Walther. Earlier local stability was known for all $a \in (0, 1)$, see [9]. As linearization fails at zero, a center manifold reduction was used. Global attractivity was proven only for $a \in (0, 0.61)$ by Garab, Kovács and Krisztin [14]. The technique of [14] worked for the more general price model

$$\dot{x}(t) = a\sum_{i=1}^{n} b_i[x(t-s_i) - x(t-r_i)] - g(x(t)). \tag{2.1.2}$$

Our proof is based on the key idea that it is possible to connect the problem with a different type of equations, namely with neutral functional differential equations, and in addition, that Lyapunov functionals can be constructed for the neutral type problems.

By using Stieltjes integrals, equations (2.1.1) and (2.1.2) can be written as

$$\dot{x}(t) = a \int_0^r x(t-s)d\eta(s) - g(x(t)), \tag{2.1.3}$$

$$\dot{y}(t) = a \int_0^r \dot{y}(t-s)d\mu(s) - g(y(t)), \tag{2.1.5}$$

assuming Hypotheses ($\mathrm{H_g}$), (H_η) and (H_μ).

In Section 2.3, we consider equation (2.1.5), formulate the hypotheses on μ, and introduce a suitable phase space. First it is shown that all solutions can be globally extended to $[-r, \infty)$. Then, in Theorem 2.3.2, a sufficient condition is given for the global asymptotic stability of the zero solution of equation (2.1.5). The proof is based on a Lyapunov functional which has been inspired by the one employed for the equation

$$\dot{x}(t) = a\dot{x}(t-1) - g(x(t)) \tag{2.1.6}$$

in the book of Kolmanovskii and Myshkis [21, Chapter 9, p. 374].

In Section 2.4, we consider equation (2.1.3) under Hypotheses ($\mathrm{H_g}$) and (H_η). Combining the global stability result of Section 2.3 for equation (2.1.5) and the continuous dependence on initial data for equation (2.1.3), the main result, that is stated as Theorem 2.4.2, is that the zero solution of equation (2.1.3) is globally asymptotically stable provided $a \in (0, 1)$. As a consequence, global asymptotic stability is obtained for the zero solution of the Erdélyi–Brunovský–Walther equation (2.1.1) and also for equation (2.1.2) for the full conjectured region $a \in (0, 1)$, see Corollaries 2.4.3, 2.4.4.

Finally in Section 2.5 we show that the global stability result for equation (2.1.3) is optimal in the sense that for $a > 1$ under the additional condition $g'(0) = 0$ the zero solution is unstable. In addition, some open problems are mentioned.

The second part of the thesis considers a system which is composed of a delay differential equation and two auxiliary equations defining the delay. The delay differential equation satisfies a negative feedback condition studied earlier in several fundamental papers [26, 27], leading to the development of topics of nonlinear functional analysis like fixed point theory in infinite dimensions. The studied particular system was introduced by Ranjan, La and Abed [31, 30] to model a rate control mechanism for a simple computer network. Mathematically, the difficulty arises from the particular form of the delay defined by the two auxiliary equations. The classical results for constant delays [12, 16], the recently developed methods for state-dependent delay [17, 35] do not seem to be applicable here. The first difficulty is to find a suitable phase space where the corresponding initial value problem has a unique maximal solution, and the solutions define a continuous semiflow. In fact, we develop two different frameworks to study the problem. These require different phase spaces and different definitions for solutions. It depends on the question which approach is more suitable. The second main result is that the rate control system of Ranjan et al. may lead to a slowly oscillating periodic rate around the optimal rate, provided that the stationary solution at the optimal rate is unstable. This answers affirmatively a conjecture of Ranjan and his coauthors [29, 28].

The network model contains a single user and a single server. The user sends data by rate $x(t)$ to the server for procession. The server processes the incoming data by the capacity c. Kelly [19] introduced the utility $U(x)$ and the price $p(x)$ per unit flow of the procession, and proposed an end user rate control algorithm as a differential equation.

As the rate $x(t)$ can be larger than the capacity of the server, the data arriving at the server may form a single waiting line (a queue) with length $y(t)$ before procession. Suppose that a unit of data, whose procession was completed and the user received an acknowledgement about it at time t, arrived at the queue $\tau(t)$ time earlier, found a queue with length $y(t-\tau(t))$, and spent waiting time $z(t) = (1/c)y(t-\tau(t))$ in the queue before its procession started. Then the model can be described by the system of equations

$$\dot{x}(t) = \kappa\big[x(t)U'(x(t)) - x(t-r_0-z(t)-r_1)p(x(t-z(t)-r_1))\big], \tag{3.1.4}$$

$$\dot{y}(t) = \begin{cases} x(t-r_0) - c & \text{if } 0 < y(t) < q, \\ [x(t-r_0)-c]^+ & \text{if } y(t) = 0, \\ -[x(t-r_0)-c]^- & \text{if } y(t) = q, \end{cases} \tag{3.1.2}$$

$$z(t) = \frac{1}{c}y(t-z(t)-r_1). \tag{3.1.3}$$

First we consider a slightly more general system of equations

$$\dot{x}(t) = F(x_t, y_t) \tag{3.1.5}$$

and (3.1.2) in $X \times Y$. The phase space $X \times Y$ contains all possible segments (x_t, y_t).

In order to see that system (3.1.4), (3.1.2), (3.1.3) is a particular case of system (3.1.5), (3.1.2) introduce $Z = [0, q/c] \subset \mathbb{R}$ as a state space for the variable $z(t)$. A cruical fact is the existence of a unique Lipschitz continuous map $\sigma : Y \to Z$ such that

$$\sigma(\psi) = \frac{1}{c}\psi(-\sigma(\psi) - r_1) \qquad (\psi \in Y).$$

Then, for a solution $(x, y) : [-r, \infty) \to \mathbb{R}^2$ of system (3.1.5), (3.1.2) in the phase space $X \times Y$, defining $z(t) = \sigma(y_t)$, $t \geq 0$, equation (3.1.3) is always satisfied for all $t \geq 0$.

Assume that a map $G : X \times Z \to \mathbb{R}$ is given such that, with the particular choice

$$F : X \times Y \ni (\varphi, \psi) \mapsto G(\varphi, \sigma(\psi)) \in \mathbb{R},$$

Hypotheses (H1)–(H4) hold. In this case system (3.1.5), (3.1.2) is equivalent to the system composed of the equations

$$\dot{x}(t) = G(x_t, z(t)), \tag{3.1.6}$$

(3.1.2) and (3.1.3). Then, in the phase space $X \times Y$, for each $(\varphi, \psi) \in X \times Y$, system (3.1.6), (3.1.2), (3.1.3) has the unique solution $x^{\varphi,\psi}[-r, \infty) \to \mathbb{R}$, $y^{\varphi,\psi} : [-r, \infty) \to \mathbb{R}$,

$z^{\varphi,\psi} : [0,\infty) \to \mathbb{R}$ where $\left(x^{\varphi,\psi}, y^{\varphi,\psi}\right)$ is the solution of system (3.1.5), (3.1.2), and $z^{\varphi,\psi}(t) = \sigma(y_t^{\varphi,\psi})$, $t \geq 0$.

In Section 3.3 we show that, under Hypotheses (H1)–(H4), for each $(\varphi, \psi) \in X \times Y$, system (3.1.5), (3.1.2) has a unique maximal solution $\left(x^{\varphi,\psi}, y^{\varphi,\psi}\right) : [-r,\infty) \to \mathbb{R}^2$. The solutions define the continuous semiflow

$$\Phi : [0,\infty) \times X \times Y \ni (t, \varphi, \psi) \mapsto \left(x_t^{\varphi,\psi}, y_t^{\varphi,\psi}\right) \in X \times Y,$$

and, for each $t \geq 0$, the solution operators $\Phi(t, \cdot, \cdot) : X \times Y \to X \times Y$ are Lipschitz continuous, see Theorem 3.3.5

We also show that system (3.1.6), (3.1.2), (3.1.3) can be studied not only in the phase space $X \times Y$, but also in $X \times Z$ with a different notion of solution. The key technical result is that there is a unique Lipschitz continuous map $\gamma : X \times Z \to Y$ so that $\psi = \gamma(\varphi, \zeta)$ satisfies $\psi(s) = c\zeta$ for $s \in [-r, -\zeta - r_1]$, and equation (3.1.2) holds a.e. in $[-\zeta - r_1, 0]$. In particular, $\zeta = (1/c)\psi(-\zeta - r_1)$. This means that the past of the length of the queue can be recovered from the past of the rate (that is $\varphi \in X$) and from the present waiting time. The maps h and k between the two different phase spaces are Lipschitz continuous, h is injective, but k is not, $k \circ h = \mathrm{id}_{X \times Z}$, and $h \circ k\big|_{h(X\times Z)} = \mathrm{id}_{h(X\times Z)}$. Theorem 3.3.11 states that for each $(\varphi, \zeta) \in X \times Z$, there exists a unique solution $x^{\varphi,\zeta} : [-r,\infty) \to \mathbb{R}$, $z^{\varphi,\zeta} : [0,\infty) \to \mathbb{R}$ of system (3.1.6), (3.1.2), (3.1.3) in the phase space $X \times Z$ satisfying the initial condition $x_0^{\varphi,\zeta} = \varphi$, $z^{\varphi,\zeta}(0) = \zeta$. Moreover,

$$\Psi : [0,\infty) \times X \times Z \ni (t, \varphi, \zeta) \mapsto \left(x_t^{\varphi,\zeta}, z^{\varphi,\zeta}(t)\right) \in X \times Z$$

is a continuous semiflow on $X \times Z$, and $\Psi(t, \varphi, \zeta) = k(\Phi(t, h(\varphi, \zeta)))$ for all $t \geq 0$.

In Section 3.4 we assume $r_0 = 0$, $r_1 = 1$ and consider system (3.1.4), (3.1.2), (3.1.3). Condition $r_0 = 0$ guarantees a single delay in equation (3.1.4), $r_1 = 1$ can be achieved by rescaling the time. Then for the new variable $v = x - x_*$ we can rewrite our system. Theorem 3.3.11 implies that system (3.1.7), (3.1.8), (3.1.9) is well posed in the phase space $\mathcal{X} \times Z$.

A solution (v, z) of system (3.1.7), (3.1.8), (3.1.9) is called *slowly oscillatory* if for any two zeros t_1, t_2 of v with $t_1 < t_2$ the inequality $z(t_2) + 1 < t_2 - t_1$ holds. This means that the distance between consecutive zeros of v is larger than the delay.

We introduce the sets W and $W_0 = W \cup \{(0,0)\}$. Then, for each $(\varphi, \zeta) \in W$, the solution $v = v^{\varphi,\zeta} : [-r,\infty) \to \mathbb{R}$, $z = z^{\varphi,\zeta} : [0,\infty) \to \mathbb{R}$ is slowly oscillatory with infinite number of zeros. The second zero t_2 of v in $(0,\infty)$ determines $t_2^* > t_2$ so that $t_2 = t_2^* - z(t_2^*) - 1$, and a return map $P : W_0 \to W_0$ can be defined. A nontrivial fixed point of P corresponds to a slowly oscillating periodic solution. A classical tool, that we apply here as well, is Browder's non-ejective fixed point theorem. A large part of Section 3.4 is devoted to the construction of a suitable subset of $\mathcal{X} \times Z$ where Browder's theorem is applicable.

It is a crucial result that $P(\varphi,\zeta)$ cannot decay too fast: there are constants $\theta > 0$, $\rho > 0$ with $v^{\varphi,\zeta}(t_2^*) \geq \theta\,(\varphi(0))^{\rho}$ for all $(\varphi,\zeta) \in W$. This fact allows to construct a proper C^2-function α. Defining the compact subsets W_{α,K_1} and W_{α,K_0} of $\mathcal{X} \times Z$, the inclusion $P(W_{\alpha,K_1}) \subseteq W_{\alpha,K_0}$ is satisfied. However, W_{α,K_1} and W_{α,K_0} are not convex. Following [25], the subset V_{α,K_1} of $C_{[-1,0]} \times \mathbb{R}$ is compact and convex. Set V_{α,K_1} can be mapped into W_{α,K_1} by the streching map Q given by $Q(\psi,\zeta) = (\varphi,\zeta)$ with $\varphi(s) = \psi(s/(\zeta+1))$, $s \in [-\zeta-1, 0]$, and $\varphi\big|_{[-r,-\zeta-1]} \equiv 0$. The squeezing map R, defined by $R(\varphi,\zeta) = (\psi,\zeta)$ with $\psi(s) = \varphi((\zeta+1)s)$, $s \in [-1,0]$, maps W_{α,K_0} into V_{α,K_1}. Browder's theorem can be applied for finding a non-ejective fixed point of the map $\Pi = R \circ P \circ Q$ in V_{α,K_1}. This yields a non-ejective fixed point of P in W_{α,K_1} as well. The non-ejective fixed point is nontrivial provided $(0,0) \in W_{\alpha,K_1}$ is ejective. Ejectivity of $(0,0) \in W_{\alpha,K_1}$ follows in a standard way from that of the zero solution of the constant delay equation $\dot{v}(t) = -f(v(t)) - g(v(t-1))$. So we can state our main result in Theorem 3.4.17.

Finally, Section 3.5 gives examples.

At the end of the thesis we summarize our results both in English and Hungarian.

Chapter 2

Global stability for price models with delay

2.1 Introduction

Our primary aim is to prove the global stability conjecture for the price model of Erdélyi, Brunovský and Walther [9, 8, 37]

$$\dot{x}(t) = a[x(t) - x(t-1)] - \beta|x(t)|x(t), \tag{2.1.1}$$

where $a > 0, \beta > 0$. They introduced equation (2.1.1) to model the short-time fluctuations of the price of a foreign currency in a domestic reference currency, although the model applies to other kind of assets as well. It is assumed that there is an equilibrium exchange rate. The deviation from the equilibrium rate is denoted by $x(t)$. The agents want to make profit from their trading, and they try to predict the future exchange rate. As they do not have precise information on the equilibrium exchange rate, for the prediction they use the movement of the exchange rate in one unit of time. That is, in case $x(t) - x(t-1) > 0$, they expect the rate to raise leading to increasing demand and thus an increase of the price. The case $x(t) - x(t-1) < 0$ is expected to lead a decreasing demand and thus a decrease of the price. This is expressed by the term $a[x(t) - x(t-1)]$. The quadratic term in equation (2.1.1) describes that once the rate moves far from its equilibrium more and more agents expect that this trend will eventually turn back.

For $0 < a < 1$, the local asymptotic stability of $x = 0$ was shown by Erdélyi, Brunovský and Walther, and they conjectured global asymptotic stability. Numerical simulations provided by Erdélyi [13] suggested the existence of a stable (slowly oscillating) periodic solution of equation (2.1.1) for $a > 1$, which was established in [9, 8]. This result has recently been generalized by Stumpf [33] for a state-dependent delay version of equation (2.1.1). Walther analyzed further the slowly oscillating periodic solution of equation (2.1.1) and showed that it converges to a square-wave solution as a tends to infinity [37], and that the period tends to infinity as $a \to 1^+$ [36].

Recently, Garab, Kovács and Krisztin [14] obtained global asymptotic stability of $x = 0$ for equation (2.1.1) provided $a \in (0, 0.61)$. The key idea of [14] to prove global asymptotic stability of $x = 0$ was to rewrite the equation as a neutral type functional differential equation. Then an equivalent equation with infinite delay was obtained for which a stability result of [22] was applied. The technique of [14] worked for the more general price model

$$\dot{x}(t) = a \sum_{i=1}^{n} b_i[x(t - s_i) - x(t - r_i)] - g(x(t)), \tag{2.1.2}$$

as well, where $a > 0$, $b_i > 0$, $0 \le s_i < r_i \le 1$, $i \in \{1, \dots, n\}$, $\sum_{i=1}^{n} b_i(r_i - s_i) = 1$ holds, and g is a smooth increasing real function with $ug(u) > 0$ for $u \neq 0$. [14] proved global asymptotic stability for equation (2.1.2) when $a \in (0, 1)$ and an additional condition was assumed, see the details in Section 2.4. In [14] it remained open to prove global asymptotic stability without the additional condition, i.e., for $a \in (0, 1)$.

In the sequel, we always assume $r > 0$, $a > 0$, and

$$(\mathrm{H_g}) \qquad \begin{cases} g : \mathbb{R} \to \mathbb{R} \text{ is } C^1\text{-smooth, } ug(u) > 0 \text{ for } u \neq 0, \\ \int_0^s g(u)\, du \to \infty \text{ as } |s| \to \infty. \end{cases}$$

By using Stieltjes integrals, equations (2.1.1) and (2.1.2) can be written as

$$\dot{x}(t) = a \int_0^r x(t - s) d\eta(s) - g(x(t)) \tag{2.1.3}$$

with η satisfying

$$(\mathrm{H}_\eta) \qquad \begin{cases} \eta : [0, r] \to [0, \infty) \text{ is of bounded variation,} \\ \eta(0) = \eta(r) = 0, \ \int_0^r \eta(s)\, ds = 1. \end{cases}$$

Following [9], $x(t)$ in equation (2.1.3) can represent the price of an asset at time t. Indeed, if $x : I \to \mathbb{R}$ is continuously differentiable on an interval containing $[t-r, t]$, then integrating the Stieltjes integral $\int_0^r x(t - s)\, d\eta(s)$ by parts, and using $\eta(0) = \eta(r) = 0$, we find

$$\begin{aligned} \int_0^r x(t - s)\, d\eta(s) &= [x(t - s)\eta(s)]_{s=0}^{s=r} - \int_0^r \eta(s)\, d_s x(t - s) \\ &= - \int_0^r \eta(s) \tfrac{d}{ds} x(t - s)\, ds \\ &= \int_0^r \dot{x}(t - s)\, d_s \left(\int_0^s \eta \right). \end{aligned} \tag{2.1.4}$$

As η is nonnegative, the function $[0, r] \ni s \mapsto \int_0^s \eta \in \mathbb{R}$ is monotone nondecreasing. Then (2.1.4) shows that the term $\int_0^r x(t - s)\, d\eta(s)$ is zero if x is constant on $[t - r, t]$, and it is positive (negative) if $\dot{x}(s) > 0$ (< 0) for all $s \in [t-r, t]$. Therefore, the term $\int_0^r x(t-s)\, d\eta(s)$ can be used to describe the tendency of the price, and the term $a \int_0^r x(t - s)\, d\eta(s)$ with

$a > 0$ can represent the positive response to the recent tendency of the price. The term $-g(x(t))$ in equation (2.1.3) is responsible for the negative feedback to the deviation of the price from the zero equilibrium.

Observe that if the function $s \mapsto \int_0^s \eta(u)du$ in the integral term $\int_0^r \dot{x}(t-s)\, d_s \left(\int_0^s \eta\right)$ in equality (2.1.4) is replaced by an arbitrary nondecreasing function $\mu : [0,r] \to \mathbb{R}$ of bounded variation, then the obtained integral term $\int_0^r \dot{x}(t-s)\, d\mu(s)$ can be still interpreted as the tendency of the price. This motivates to study the neutral type differential equation

$$\dot{y}(t) = a \int_0^r \dot{y}(t-s) d\mu(s) - g(y(t)), \tag{2.1.5}$$

as well as a price model provided $a > 0$ and $\mu : [0,r] \to \mathbb{R}$ is of bounded variation and nondecreasing with an additional technical assumption given in Section 2.3.

There is another reason to study the neutral type equation (2.1.5). It plays a crucial role in the proof of the stability results for equations (2.1.1), (2.1.2), (2.1.3). However, equation (2.1.3) and equation (2.1.5) are not equivalent. A solution of equation (2.1.3) satisfies equation (2.1.5) with $\mu(s) = \int_0^s \eta$ only for $t > r$. The phase spaces and the stability definitions are also different for equations (2.1.3) and (2.1.5).

The chapter is organized as follows. In Section 2.3, we consider equation (2.1.5), formulate the hypotheses on μ, and introduce a suitable phase space. First it is shown that all solutions can be globally extended to $[-r,\infty)$. Then a sufficient condition is given for the global asymptotic stability of the zero solution of equation (2.1.5). The proof is based on a Lyapunov functional which has been inspired by the one employed for the equation

$$\dot{x}(t) = a\dot{x}(t-1) - g(x(t)) \tag{2.1.6}$$

in the book of Kolmanovskii and Myshkis [21, Chapter 9, p. 374]. There, equation (2.1.6) describes a shunted power transmission line.

In Section 2.4, we consider equation (2.1.3) under Hypotheses ($\mathrm{H_g}$) and (H_η). Combining the global stability result of Section 2.3 for equation (2.1.5) and the continuous dependence on initial data for equation (2.1.3), the main result is that the zero solution of equation (2.1.3) is globally asymptotically stable provided $a \in (0,1)$. As a consequence, global asymptotic stability is obtained for the zero solution of the Erdélyi–Brunovský–Walther equation (2.1.1) and also for equation (2.1.2) for the full conjectured region $a \in (0,1)$.

Finally in Section 2.5 we show that the global stability result for equation (2.1.3) is optimal in the sense that for $a > 1$ under the additional condition $g'(0) = 0$ the zero solution is unstable. In addition, some open problems are mentioned.

2.2 Preliminary results

In the theory of neutral differential equations, we follow Kolmanovskii and Myshkis [21]. Note that there is also a bit different approach by Hale and Verduyn Lunel [16].

We consider the neutral differential equation

$$\dot{x}(t) = f(t, x_t, \dot{x}_t) \tag{2.2.1}$$

with initial condition

$$x_{t_0} = \psi. \tag{2.2.2}$$

The next result is Theorem 3.1 on page 107 in [21].

Theorem A. *Let* $E = [t_0, \infty) \times C([-r, 0], \mathbb{R})^2$, $f : E \to \mathbb{R}$ *be a continuous functional and in some neighbourhood of any point of* E *it satisfies the condition*

$$|f(t, \psi^1, \chi^1) - f(t, \psi^2, \chi^2)| \leq L\|\psi^1 - \psi^2\|_{[-r,0]} + l\|\chi^1 - \chi^2\|_{[-r,0]}$$

with constants $L \in [0, \infty)$, $l \in [0, 1)$ *(which may depend on the point). Assume also that* $\psi \in C^1([-r, 0], \mathbb{R})$ *and the sewing condition*

$$\dot{\psi}(0) = f(t_0, \psi, \dot{\psi})$$

is fulfilled. Then there exists a constant $t_\psi \in (t_0, \infty]$ *such that*

a) there exists a solution x *of* (2.2.1), (2.2.2) *on the interval* $[t_0, t_\psi)$*;*

b) on any interval $[t_0, t_1] \subset [t_0, t_\psi)$ *this solution is unique;*

c) if $t_\psi < \infty$ *then* $\dot{x}(t)$ *does not have a finite limit as* $t \to t_\psi^-$*;*

d) the solution x *and* $\dot{x}$ *depend continuously on* f, ψ*.*

2.3 Global stability in equation (2.1.5)

In this section we study equation (2.1.5) under condition $a \in (0, 1)$, Hypothesis ($\mathrm{H_g}$), and the assumption on μ descibed below.

First we define a step function with (possibly) infinite number of steps. Let $(c_n)_{n=0}^\infty$ be a sequence of nonnegative numbers with $\sum_{n=0}^\infty c_n \leq 1$, and let $(r_n)_{n=0}^\infty$ be a sequence in $[0, r]$ such that $r_0 = 0$, and $r_n > 0$ for all $n \in \mathbb{N}$. Let $H : [0, r] \to \mathbb{R}$ be given by $H(0) = 0$, $H(s) = 1$ for $s \in (0, r]$. Define $\sigma : [0, r] \to \mathbb{R}$ by

$$\sigma(s) = c_0 H(s) + \sum_{n:\ r_n \leq s} c_n, \quad s \in [-r, 0].$$

Let a nondecreasing and absolutely continuous $\nu : [0, r] \to \mathbb{R}$ be given with $\nu(r) - \nu(s) \leq 1$.

Our hypothesis on μ is that it is nondecreasing without a singular part, that is,

$$(\mathrm{H}_\mu) \quad \begin{cases} \mu : [0, r] \to \mathbb{R} \text{ is given by } \mu = \nu + \sigma \\ \text{such that } \int_0^r d\mu = 1, \text{ i.e., } \nu(r) - \nu(0) + \sum_{n=0}^{\infty} c_n = 1 \end{cases}$$

holds.

Define the set

$$Y = \left\{ \psi \in C^1([-r, 0], \mathbb{R}) \;\middle|\; \dot{\psi}(0) = a \int_0^r \dot{\psi}(-s)\, d\mu(s) - g(\psi(0)) \right\},$$

and let

$$\|\psi\|_Y = \left((\psi(0))^2 + \int_0^r (\dot{\psi}(-s))^2\, ds \right)^{1/2}$$

for $\psi \in Y$. Set Y will be the phase space for equation (2.1.5).

A solution of equation (2.1.5) with initial function $\psi \in Y$ is a continuously differentiable function $y = y^\psi : [-r, t_\psi) \to \mathbb{R}$ such that $y_0 = \psi$, and equation (2.1.5) holds for all $t \in (0, t_\psi)$. The solution y^ψ is called a maximal solution if any other solution with the same initial function is a restriction of y^ψ.

From $g(0) = 0$ it is clear that $y = 0$ is a solution of (2.1.5), and by $(\mathrm{H_g})$ it is the only equilibrium solution. The solution $y = 0$ of equation (2.1.5) is called stable if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that, for each $\psi \in Y$ with $\|\psi\|_Y < \delta(\varepsilon)$, the solution y^ψ exists on $[-r, \infty)$ and $\|y_t^\psi\|_Y < \varepsilon$ for all $t \geq 0$. The solution $y = 0$ is called globally asymptotically stable if it is stable and for each $\psi \in Y$ the solution y^ψ exists on $[-r, \infty)$ and $\|y_t^\psi\|_Y \to 0$ as $t \to \infty$.

Theorem A states that for each $\psi \in Y$, equation (2.1.5) has a unique maximal solution $y^\psi : [-r, t_\psi) \to \mathbb{R}$, and in case $t_\psi < \infty$ the finite limit $\lim_{t \to t_\psi -} \dot{y}^\psi(t)$ does not exist. We will use this result to show that for any $\psi \in Y$ there exists a unique solution on $[-r, \infty)$.

Proposition 2.3.1. *Assume Hypotheses* $(\mathrm{H_g})$, (H_μ) *hold, and* $a \in (0, 1)$. *Let* $\psi \in Y$ *and consider the unique maximal solution* $y^\psi : [-r, t_\psi) \to \mathbb{R}$ *of equation* (2.1.5). *If* y^ψ *is bounded on* $[-r, t_\psi)$ *then* $t_\psi = \infty$.

Proof. Let $\psi \in Y$, $y = y^\psi : [-r, t_\psi) \to \mathbb{R}$, and let y be bounded on $[-r, t_\psi)$.

Assume $t_\psi < \infty$. Then, by Theorem A, the finite limit $\lim_{t \to t_\psi -} \dot{y}(t)$ does not exist.

First we show that $\dot{y}$ is bounded on $[-r, t_\psi)$. If $\dot{y}$ is unbounded from above on $[-r, t_\psi)$ then we can choose a sequence $(\tau_n)_{n=1}^\infty$ in $[0, t_\psi)$ such that $\tau_n \to t_\psi$, $\dot{y}(\tau_n) \to \infty$ as $n \to \infty$, $\dot{y}(t) < \dot{y}(\tau_n)$ for all $t \in [-r, \tau_n)$, $n \in \mathbb{N}$. For arbitrary $n \in \mathbb{N}$, by using Hypothesis (H_μ), we have

$$\begin{aligned}\dot{y}(\tau_n) &= a\int_0^r \dot{y}(\tau_n - s)\,d\mu(s) - g(y(\tau_n))\\ &\le a\int_0^r \dot{y}(\tau_n)\,d\mu(s) - g(y(\tau_n)) = a\dot{y}(\tau_n) - g(y(\tau(t_n))).\end{aligned}$$

Hence, from $a \in (0,1)$ and $\dot{y}(\tau_n) \to \infty$, it follows that

$$-g(y(\tau_n)) \ge (1-a)\dot{y}(\tau_n) \to \infty \quad \text{as } n \to \infty.$$

As y is bounded, this is a contradiction. The case when $\dot{y}$ is unbounded from below leads similarly to a contradiction. Thus, $\dot{y}$ is bounded on $[-r, t_\psi)$.

Define

$$\alpha = \liminf_{t \to t_\psi^-} \dot{y}(t), \quad \beta = \limsup_{t \to t_\psi^-} \dot{y}(t).$$

We know that $-\infty < \alpha < \beta < \infty$. There are strictly increasing sequences $(t_n)_{n=1}^\infty, (s_n)_{n=1}^\infty$ in $[0, t_\psi)$ such that $t_n \to t_\psi$, $s_n \to t_\psi$, and

$$\lim_{n\to\infty} \dot{y}(s_n) = \alpha, \quad \lim_{n\to\infty} \dot{y}(t_n) = \beta.$$

Choose $\alpha' < \alpha < \beta < \beta'$ so that $a(\beta' - \alpha') < \beta - \alpha$. There exists a $\delta > 0$ such that $\dot{y}(t) \in [\alpha', \beta']$ for all $t \in [t_\psi - 2\delta, t_\psi)$. From (2.1.5) it follows that

$$\begin{aligned}\dot{y}(t_n) - \dot{y}(s_n) &= a\int_0^\delta (\dot{y}(t_n - s) - \dot{y}(s_n - s))\,d\mu(s)\\ &\quad + a\int_\delta^r (\dot{y}(t_n - s) - \dot{y}(s_n - s))\,d\mu(s) - g(y(t_n)) + g(y(s_n)).\end{aligned}$$

We have

$$\lim_{n\to\infty} a\int_\delta^r (\dot{y}(t_n - s) - \dot{y}(s_n - s))\,d\mu(s) = 0$$

because $\dot{y}$ is uniformly continuous on $[-r, t_\psi - \delta]$. In addition,

$$\lim_{n\to\infty} [g(y(t_n)) - g(y(s_n))] = 0$$

since the boundedness of $\dot{y}$ on $[-r, t_\psi)$ implies the uniform continuity of y and $g \circ y$, and that $g(y(t))$ has a finite limit at t_ψ. Combining these facts with $\dot{y}(t_n) - \dot{y}(s_n) \to \beta - \alpha$ as $n \to \infty$, one obtains

$$\begin{aligned}\beta - \alpha = \lim_{n\to\infty} (\dot{y}(t_n) - \dot{y}(s_n)) &= \lim_{n\to\infty} a\int_0^\delta (\dot{y}(t_n - s) - \dot{y}(s_n - s))\,d\mu(s)\\ &\le a\int_0^\delta (\beta' - \alpha')\,d\mu(s) \le a(\beta' - \alpha'),\end{aligned}$$

a contradiction. Therefore, $t_\psi = \infty$, and the proof is complete. □

Next we show the global asymptotic stability of the zero solution of equation (2.1.5).

Theorem 2.3.2. *Assume Hypotheses* $(\mathrm{H_g})$, (H_μ) *hold, and* $a \in (0,1)$. *Then for each* $\psi \in Y$ *the unique maximal solution* y^ψ *of equation* (2.1.5) *is defined on* $[-r,\infty)$, *and the zero solution of* (2.1.5) *is globally asymptotically stable.*

Proof. Define the function

$$K : [0,r] \ni s \mapsto \int_s^r d\mu \in [0,1].$$

According to Hypothesis (H_μ), let $K_1(s) = \int_s^r d\nu$ and $K_2(s) = \int_s^r d\sigma$. Then $K(s) = K_1(s) + K_2(s)$, and

$$K_1(s) = \nu(r) - \nu(s), \quad K_2(s) = \begin{cases} \sum_{n=0}^{\infty} c_n & \text{for } s = 0, \\ \sum_{r_n > s} c_n & \text{for } s \in (0,r]. \end{cases}$$

Let $\psi \in Y$ and consider the unique solution $y = y^\psi : [-r, t_\psi) \to \mathbb{R}$. For $t \in [0, t_\psi)$, define

$$\begin{aligned} w(t) &= \int_{t-r}^{t} K(t-s)(\dot{y}(s))^2\,ds + \frac{2}{a^2}\int_0^{y(t)} g(u)\,du \\ &= \int_{t-r}^{t} \left[K_1(t-s) + K_2(t-s)\right](\dot{y}(s))^2\,ds + \frac{2}{a^2}\int_0^{y(t)} g(u)\,du. \end{aligned}$$

As y and K_1 are continuously differentiable functions, the map

$$[0, t_\psi) \ni t \mapsto \int_{t-r}^{t} K_1(t-s)(\dot{y}(s))^2\,ds \in \mathbb{R}$$

is continuously differentiable, and

$$\begin{aligned} &\frac{d}{dt}\int_{t-r}^{t} K_1(t-s)(\dot{y}(s))^2\,ds \\ &= K_1(0)(\dot{y}(t))^2 - K_1(r)(\dot{y}(t-r))^2 + \int_{t-r}^{t} \frac{d}{dt}K_1(t-s)(\dot{y}(s))^2\,ds \\ &= \left[\nu(r) - \nu(0)\right](\dot{y}(t))^2 + \int_0^r (\dot{y}(t-s))^2 K_1'(s)\,ds \\ &= \left[\nu(r) - \nu(0)\right](\dot{y}(t))^2 - \int_0^r (\dot{y}(t-s))^2\,d\nu(s). \end{aligned}$$

Observe that

$$\int_{t-r}^{t} K_2(t-s)(\dot{y}(s))^2\,ds = \sum_{n=0}^{\infty} c_n \int_{t-r_n}^{t} (\dot{y}(s))^2\,ds.$$

This series of functions is continuously differentiable, it can be differentiated term by term, and

$$\frac{d}{dt}\int_{t-r}^{t} K_2(t-s)(\dot y(s))^2\,ds = \sum_{n=0}^{\infty} c_n\left[(\dot y(t))^2 - (\dot y(t-r_n))^2\right]$$
$$= \left(\sum_{n=0}^{\infty} c_n\right)(\dot y(t))^2 - \int_0^r (\dot y(t-s))^2\,d\sigma(s).$$

The last term in $w(t)$ is clearly continuously differentiable with

$$\frac{d}{dt}\frac{2}{a^2}\int_0^{y(t)} g(u)\,du = \frac{2}{a^2}g(y(t))\dot y(t).$$

Therefore, w is continuously differentiable on $[0, t_\psi)$, and

$$w'(t) = \left[\nu(r) - \nu(0) + \sum_{n=0}^{\infty} c_n\right](\dot y(t))^2 - \int_0^r (\dot y(t-s))^2\,d(\nu+\sigma)(s) + \frac{2}{a^2}g(y(t))\dot y(t).$$

By Hypothesis (H_μ), we have $\nu(r)-\nu(0)+\sum_{n=0}^{\infty} c_n = 1$, and $\mu = \nu+\sigma$. Jensen's inequality implies

$$\left(\int_0^r \dot y(t-s)\,d\mu(s)\right)^2 \le \int_0^r (\dot y(t-s))^2\,d\mu(s).$$

Combining the above relations, it follows that

$$w'(t) \le (\dot y(t))^2 - \left(\int_0^r \dot y(t-s)\,d\mu(s)\right)^2 + \frac{2}{a^2}g(y(t))\dot y(t). \tag{2.3.1}$$

From equation (2.1.5), the term $\int_0^r \dot y(t-s)\,d\mu(s)$ is equal to $(1/a)[\dot y(t)+g(y(t))]$. Therefore, by (2.3.1),

$$w'(t) \le -\left(\frac{1}{a^2}-1\right)(\dot y(t))^2 - \frac{1}{a^2}(g(y(t)))^2 \tag{2.3.2}$$

holds for all $t \in [0, t_\psi)$.

From inequality (2.3.2), by $a \in (0,1)$, it follows that w is a nonincreasing function on $[0, t_\psi)$, and $w(t) \in [0, w(0)]$ for all $t \in [0, t_\psi)$. This fact and the definition of w gives

$$\int_0^{y(t)} g(u)\,du \in \left[0, \frac{a^2}{2}w(0)\right]$$

for all $t \in [0, t_\psi)$. By Hypothesis ($\mathrm{H_g}$) we obtain that y is bounded on $[0, t_\psi)$, and then on $[-r, t_\psi)$. Proposition 2.3.1 can be applied to conclude $t_\psi = \infty$.

Thus, inequality (2.3.2) holds for all $t \in [0,\infty)$. Then there exists $w_* \ge 0$ such that $w(t) \to w_*$ as $t \to \infty$, and, for each $T \ge 0$,

$$w(0) - w_* \ge w(0) - w(T) = -\int_0^T w'(t)\,dt$$
$$\ge \left(\frac{1}{a^2}-1\right)\int_0^T (\dot y(t))^2\,dt + \frac{1}{a^2}\int_0^T (g(y(t)))^2\,dt.$$

Hence it follows that

$$\int_0^\infty (\dot{y}(t))^2\,dt \le \frac{a^2}{1-a^2} w(0), \tag{2.3.3}$$

$$\int_0^\infty (g(y(t)))^2\,dt \le a^2 w(0). \tag{2.3.4}$$

In particular, (2.3.3) implies

$$\int_{t-r}^t (\dot{y}(s))^2\,ds \to 0 \quad \text{as } t \to \infty.$$

Then, using $K(s) \in [0,1]$, $s \in [0,r]$, one finds that

$$\int_0^r K(s)(\dot{y}(t-s))^2\,ds \le \int_0^r (\dot{y}(t-s))^2\,ds = \int_{t-r}^t (\dot{y}(s))^2\,ds,$$

and

$$\int_{t-r}^t K(t-s)(\dot{y}(s))^2\,ds = \int_0^r K(s)(\dot{y}(t-s))^2\,ds \to 0 \quad \text{as } t \to \infty.$$

Therefore,

$$w_* = \lim_{t\to\infty} w(t) = \frac{2}{a^2} \lim_{t\to\infty} \int_0^{y(t)} g(u)\,du.$$

From Condition (H_g), the map $[0,\infty) \ni s \mapsto \int_0^s g(u)\,du \in \mathbb{R}$ strictly increases from 0 to ∞, and the map $(-\infty,0] \ni s \mapsto \int_0^s g(u)\,du \in \mathbb{R}$ strictly decreases from ∞ to 0. Consequently, there exists $y_* \in \mathbb{R}$ so that $y(t) \to y_*$ as $t \to \infty$. By (2.3.4), the integral $\int_0^\infty (g(y(t)))^2\,dt$ converges. These facts combined yield $y_* = 0$. Thus, $y^\psi(t) \to 0$ as $t \to \infty$ for all $\psi \in Y$.

In order to show local stability of the zero solution, let $\varepsilon >$ be given. By Hypothesis (H_g), there exists $m \ge a^2$ such that

$$|g(u)| \le m|u| \quad \text{for all } u \in [-1,1].$$

Choose $\delta = \delta(\varepsilon) \in (0,1)$ so that

$$\left(1 + \frac{m}{1-a^2}\right)\delta^2 < \frac{\varepsilon^2}{2}. \tag{2.3.5}$$

In addition, by (H_g), δ can be chosen so small that

$$\int_0^s g(u)\,du < \frac{m}{2}\delta^2 \quad \text{implies} \quad s^2 < \frac{\varepsilon^2}{2}. \tag{2.3.6}$$

Let $\psi \in Y$ with $\|\psi\|_Y < \delta$, and let $y = y^\psi : [-r,\infty) \to \mathbb{R}$ be the corresponding solution of (2.1.5). Then $K(s) \in [0,1]$, $s \in [0,r]$, $|\psi(0)| \le \|\psi\|_Y < \delta < 1$, and the choice of m guarantee that

$$\begin{aligned} w(0) &= \int_{-r}^0 K(-s)(\dot{\psi}(s))^2\,ds + \frac{2}{a^2}\int_0^{\psi(0)} g(u)\,du \le \int_{-r}^0 (\dot{\psi}(s))^2\,ds + \frac{2}{a^2}\frac{m}{2}(\psi(0))^2 \\ &\le \frac{m}{a^2}\left(\int_{-r}^0 (\dot{\psi}(s))^2\,ds + (\psi(0))^2\right) = \frac{m}{a^2}\|\psi\|_Y^2 < \frac{m}{a^2}\delta^2. \end{aligned}$$

This estimation for $w(0)$ combined with inequality (2.3.3) yields

$$\int_0^\infty (\dot{y}(t))^2\, dt < \frac{m}{1-a^2}\delta^2. \tag{2.3.7}$$

The estimation $w(0) < (m/a^2)\delta^2$, the definition of $w(t)$, and $w(t) \le w(0)$ combined give

$$\int_0^{y(t)} g(u)\, du \le \frac{a^2}{2} w(t) \le \frac{a^2}{2} w(0) < \frac{m}{2}\delta^2, \quad t \ge 0. \tag{2.3.8}$$

From $\|\psi\|_Y < \delta$ and inequality (2.3.7) it follows that

$$\int_{-r}^0 (\dot{y}(t+s))^2\, ds < \left(1 + \frac{m}{1-a^2}\right)\delta^2 \quad \text{for all } t \ge 0. \tag{2.3.9}$$

A combination of (2.3.9), (2.3.5), (2.3.6), (2.3.8) implies

$$\|y_t\|_Y = \left(\int_{-r}^0 (\dot{y}(t+s))^2\, ds + (y(t))^2\right)^{1/2} < \left(\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}\right)^{1/2} = \varepsilon$$

for all $t \ge 0$. This proves the local stability of $y = 0$.

The global attractivity of $y = 0$, that is $\|y_t^\psi\|_Y \to 0$ as $t \to \infty$, for all $\psi \in Y$, follows from $\int_{-r}^0 (\dot{y}^\psi(t+s))^2\, ds \to 0$, $t \to \infty$, implied by (2.3.3), and $y^\psi(t) \to 0$, $t \to \infty$. Therefore, $y = 0$ is globally asymptotically stable. □

2.4 Global stability in equation (2.1.3)

In this section, we consider equation (2.1.3) under Hypotheses ($\mathrm{H_g}$) and (H_η).

The natural phase space for equation (2.1.3) is $C([-r,0],\mathbb{R})$. A maximal solution of (2.1.3) with initial function $\varphi \in C([-r,0],\mathbb{R})$ is a continuous function $x = x^\varphi : [-r, t_\varphi) \to \mathbb{R}$ with $t_\varphi > 0$ so that $x|_{[-r,0]} = \varphi$, x is differentiable on $(0, t_\varphi)$, equation (2.1.3) holds on $(0, t_\varphi)$, and any other solution with the same initial function is a restriction of x^φ.

Recall that the solution $x = 0$ of equation (2.1.3) is stable if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that, for each $\varphi \in C([-r,0],\mathbb{R})$ with $\|\varphi\| < \delta(\varepsilon)$, the solution x^φ exists on $[-r,\infty)$ and $\|x_t^\varphi\| < \varepsilon$ for all $t \ge 0$. The solution $x = 0$ is globally asymptotically stable if in addition to stability for each $\varphi \in C([-r,0],\mathbb{R})$ the solution x^φ exists on $[-r,\infty)$ and $\|x_t^\varphi\| \to 0$ as $t \to \infty$.

First, for arbitrary $a > 0$, we show that the maximal solutions exist on $[-r,\infty)$.

Proposition 2.4.1. *Assume that $a > 0$ and Conditions ($\mathrm{H_g}$), (H_η) hold. For each $\varphi \in C([-r,0],\mathbb{R})$ equation (2.1.3) has a unique maximal solution x^φ which is defined on $[-r,\infty)$.*

Proof. The map

$$f : C([-r,0],\mathbb{R}) \ni \varphi \mapsto a \int_0^r \varphi(-s)\, d\eta(s) - g(\varphi(0)) \in \mathbb{R}$$

is continuous, it is also Lipschitzian in each compact subset of $C([-r,0],\mathbb{R})$, and f takes bounded sets into bounded sets. Then, by Theorem [16, Chapter 2, Theorem 2.3], for each $\varphi \in C([-r,0],\mathbb{R})$ there is a unique maximal solution $x^\varphi : [-r,t_\varphi) \to \mathbb{R}$ of equation (2.1.3). Moreover, by Theorem [16, Chapter 2, Theorem 3.2], in case $t_\varphi < \infty$ we have $\|x_t^\varphi\| \to \infty$ as $t \to t_\varphi^-$.

Let $\varphi \in C([-r,0],\mathbb{R})$, and let $|\eta|$ denote the total variation of η. Define

$$k(t) = (\|\varphi\| + 1)\, e^{(a|\eta|+1)t}, \quad t \in [0,\infty).$$

We claim that

$$|x^\varphi(t)| < k(t) \quad \text{for all } t \in [0,t_\varphi). \tag{2.4.1}$$

If inequality (2.4.1) does not hold then, by $|x^\varphi(0)| \le \|\varphi\| < k(0)$, there exists $t_0 \in (0,t_\varphi)$ such that $|x^\varphi(t)| < k(t)$ for all $t \in [0,t_0)$, $|x^\varphi(t_0)| = k(t_0)$, $|\dot{x}^\varphi(t_0)| \ge k'(t_0) = (a|\eta| + 1)k(t_0)$. Assume $x^\varphi(t_0) = k(t_0)$ (the case $-x^\varphi(t_0) = k(t_0)$ is similar). Then equation (2.1.3), $k(t) < k(t_0)$ for $t \in [0,t_0)$, and $g(x^\varphi(t_0)) > 0$ combined yield the contradiction

$$\dot{x}^\varphi(t_0) \le a|\eta|k(t_0) - g(x^\varphi(t_0)) < a|\eta|k(t_0).$$

Therefore, inequality (2.4.1) holds. Then, by Theorem [16, Chapter 2, Theorem 3.2], $t_\varphi = \infty$. □

Now we consider equation (2.1.3) for $a \in (0,1)$, and prove the global asymptotic stability of the zero solution.

Theorem 2.4.2. *Assume Hypotheses* $(\mathrm{H_g})$, (H_η) *hold, and* $a \in (0,1)$. *Then the zero solution of equation* (2.1.3) *is globally asymptotically stable.*

Proof. In order to show local stability, let $\varepsilon > 0$ be given. By Theorem 2.3.2 there exists $\gamma = \gamma(\varepsilon) > 0$ such that for each $\psi \in Y$ with $\|\psi\|_Y < \gamma$, for the solution y^ψ of equation (2.1.5), the inequality $\|y_t^\psi\|_Y < \varepsilon$ holds for all $t \ge 0$.

Define

$$\delta_1 = \frac{1}{\sqrt{1+r}}\gamma.$$

Let $|\eta|$ denote the total variation of η. By Condition $(\mathrm{H_g})$, we can find $\delta_2 \in (0,\delta_1)$ such that

$$(1 + a|\eta|)\delta_2 + \max_{|u|\le\delta_2} |g(u)| < \delta_1.$$

By continuous dependence on initial data of solutions of equation (2.1.3), see Theorem [16, Chapter 2, Theorem 2.2], we can choose $\delta > 0$ such that, for each $\varphi \in C([-r,0],\mathbb{R})$ with $\|\varphi\| < \delta$, the unique solution x^φ of equation (2.1.3) satisfies

$$|x^\varphi(t)| < \min\{\varepsilon, \delta_2\} \quad \text{for all } t \in [-r, r].$$

Then for x^φ with $\varphi \in C([-r,0],\mathbb{R})$ and $\|\varphi\| < \delta$, from equation (2.1.3) it follows that

$$|\dot{x}^\varphi(t)| \le a|\eta|\delta_2 + \max_{|u|\le\delta_2} |g(u)| < \delta_1 \quad \text{for all } t \in (0, r]$$

and

$$|x^\varphi(t)| < \delta_2 < \delta_1 \quad \text{for all } t \in [0, r].$$

By the uniform continuity of $x^\varphi|_{[-r,r]}$, there exists the limit

$$\lim_{t\to 0^+} \dot{x}^\varphi(t) = a \int_0^r \varphi(-s)\, d\eta(s) - g(\varphi(0)).$$

It follows that x^φ is right differentiable at $t = 0$, and $x^\varphi_r \in C^1([-r,0],\mathbb{R})$. Then $x^\varphi_r \in Y$ and

$$\|x^\varphi_r\|_Y = \left(\int_0^r (\dot{x}^\varphi(t))^2\, dt + (x^\varphi(t))^2\right)^{1/2} < \left(r\delta_1^2 + \delta_1^2\right)^{1/2} = \sqrt{r+1}\delta_1 = \gamma.$$

By (2.1.4), $y(t) = x^\varphi(t+r)$, $t \in [-r, \infty)$, is a solution of equation (2.1.5) with $\mu(s) = \int_0^s \eta$ and initial function $y_0 = x^\varphi_r \in Y$. Then the choice of γ guarantees that

$$\|y_t\|_Y = \|x^\varphi_{t+r}\|_Y < \varepsilon \quad \text{for all } t \ge 0.$$

The definition of $\|\cdot\|_Y$ and the choice of δ imply that for each $\varphi \in C([-r,0],\mathbb{R})$ with $\|\varphi\| < \delta$, the inequality

$$|x^\varphi(t)| < \varepsilon \quad \text{for all } t \ge 0$$

holds. Therefore the zero solution of equation (2.1.3) is locally stable.

Global attractivity of $x = 0$ also follows from Theorem 2.3.2 since $x^\varphi_r \in Y$ for all $\varphi \in C([-r,0],\mathbb{R})$. □

equation (2.1.1) is a particular case of (2.1.3) with $r = 1$, $g(u) = \beta|u|u$ and

$$\eta(s) = \begin{cases} 1 & \text{if } s \in (0,1), \\ 0 & \text{if } s = 0 \text{ or } s = 1. \end{cases}$$

Therefore, Theorem 2.4.2 implies a solution to the global stability conjecture of [9, 8, 37].

Corollary 2.4.3. *If $a \in (0,1)$ then the zero solution of equation* (2.1.1) *is globally asymptotically stable.*

Let the constants $b_i > 0$, $0 \le s_i < r_i \le 1$, $i \in \{1, \dots, n\}$, be given so that $\sum_{i=1}^n b_i(r_i - s_i) = 1$ holds. Define the functions

$$\eta_i(s) = \begin{cases} b_i & \text{if } s \in (s_i, r_i), \\ 0 & \text{if } s \in [0, s_i] \cup [r_i, 1], \end{cases}$$

for $i \in \{1, \dots, n\}$, and let

$$\eta : [0,1] \ni s \mapsto \sum_{i=1}^n \eta_i(s) \in \mathbb{R}.$$

Then it is easy to see that η satisfies Condition (H_η) with $r = 1$, and equation (2.1.2) is a particular case of equation (2.1.3). Therefore the following result holds.

Corollary 2.4.4. *If $a \in (0,1)$, $b_i > 0$, $0 \le s_i < r_i \le 1$, $\sum_{i=1}^n b_i(r_i - s_i) = 1$, and g satisfies Condition* ($\mathrm{H_g}$)*, then the zero solution of equation* (2.1.2) *is globally asymptotically stable.*

The equation

$$\dot{x}(t) = \alpha \sum_{i=1}^n \beta_i [x(t - s_i) - x(t - r_i)] - g(x(t)), \tag{2.4.2}$$

was studied in [14] under the conditions $\alpha > 0$, $\beta_i > 0$, $0 \le s_i < r_i \le 1$, $i \in \{1, \dots, n\}$, $\sum_{i=1}^n \beta_i = 1$, and g satisfied a condition stronger than ($\mathrm{H_g}$). Setting

$$a = \alpha \sum_{i=1}^n \beta_i (r_i - s_i), \qquad b_i = \frac{\beta_i}{\sum_{i=1}^n \beta_i (r_i - s_i)} \quad (i \in \{1, \dots, n\}),$$

it is clear that equation (2.4.2) is equivalent to equation (2.1.2). Consequently, we obtain the following result.

Corollary 2.4.5. *If $\alpha > 0$, $\beta_i > 0$, $0 \le s_i < r_i \le 1$, $i \in \{1, \dots, n\}$, and g satisfies* ($\mathrm{H_g}$)*, then*

$$\alpha \sum_{i=1}^n \beta_i (r_i - s_i) < 1$$

implies the global asymptotic stability of the zero solution of equation (2.4.2)*.*

We remark that [14] proved global asymptotic stability of $x = 0$ for equation (2.4.2) assuming $\alpha \sum_{i=1}^n \beta_i (r_i - s_i) < 1$, a condition on g that is stronger than ($\mathrm{H_g}$), and the extra condition

$$\alpha^2 \sum_{i=1}^n \beta_i (r_i^2 - s_i^2) < \left(1 - \alpha \sum_{i=1}^n \beta_i (r_i - s_i)\right)^2$$

was also used. By the local stability result for equation (2.4.2), and by the analogous conjecture for equation (2.1.1), it was suspected in [14] that Corollary 2.4.5 holds.

2.5 Discussion

In this section we show that the global stability result $a < 1$ for equation (2.1.3), and then also for equations (2.1.1) and (2.1.2), is sharp in the sense that under the additional condition $g'(0) = 0$ inequality $a > 1$ implies that the zero solution is unstable. Remark that $g'(0) = 0$ holds for equation (2.1.1). Paper [14] also assumed $g'(0) = 0$ when studied equation (2.4.2) or equivalenty equation (2.1.2).

Theorem 2.5.1. *Suppose that Hypotheses* $(\mathrm{H_g})$, (H_η) *hold, and* $g'(0) = 0$. *If* $a > 1$ *then the zero solution of equation* (2.1.3) *is unstable.*

Proof. By $g'(0) = 0$, the linear variational equation of equation (2.1.3) is

$$\dot{x}(t) = a \int_0^r x(t-s)\, d\eta(s). \tag{2.5.1}$$

The characteristic function is $\Delta : \mathbb{C} \ni \lambda \mapsto \lambda - a \int_0^r e^{-\lambda s}\, d\eta(s) \in \mathbb{C}$.

Condition (H_η) and integration by parts for Stieltjes integrals gives

$$\int_0^r e^{-\lambda s}\, d\eta(s) = \left[e^{-\lambda s}\eta(s)\right]_{s=0}^{s=r} - \int_0^r \eta(s)\, d_s\left(e^{-\lambda s}\right) = \lambda \int_0^r \eta(s) e^{-\lambda s}\, ds.$$

Hence, if λ is real and tends to ∞, then, using again (H_η), it is clear that

$$\Delta(\lambda) = \lambda - a \int_0^r e^{-\lambda s}\, d\eta(s) = \lambda \left[1 - a \int_0^r \eta(s) e^{-\lambda s}\, ds\right] \to \infty.$$

Combining this fact with $\Delta(0) = -a \int_0^r d\eta = 0$ and $\Delta'(0) = 1 - a \int_0^r \eta(s)\, ds = 1 - a < 0$, it follows that Δ has a real positive zero. Therefore, a classical result, e.g. from [16], yields that $x = 0$ is unstable. □

Equations (2.1.1), (2.1.2), (2.1.3) as price models are also important when the zero solution is unstable. In this case there are results about the dynamics only for equation (2.1.1) with a single delay, see [9, 8, 37, 36]. It is an interesting open problem to understand the dynamics in the presence of multiple or distributed delays, that is, for equations (2.1.2) and (2.1.3).

We have seen in the Introduction that the neutral differential equation (2.1.5) is also interesting as a price model. Global asymptotic stability is obtained in this chapter for $a \in (0, 1)$ provided $(\mathrm{H_g})$ and (H_μ) hold. However, the understanding of the dynamics is completely open for $a \geq 1$. The simple-looking equation (2.1.6) is a particular case. There are results only for $a \in (0, 1)$, and the case $a \geq 1$ is also an interesting open problem.

Chapter 3

A differential equation with a state-dependent queueing delay

3.1 Introduction

We consider a system which is composed of a delay differential equation and auxiliary equations defining the delay. The delay differential equation satisfies a negative feedback condition analogously to earlier works by Mallet-Paret and Nussbaum [26, 27], Arino, Hadeler, Hbid and Magal [2, 25], Krisztin and Arino [23], Walther [39, 41, 38, 35]. In [26, 35] the state-dependent delay was an explicitly given function (i.e., no auxiliary equation). Walther [39, 38] studied problems where the state-dependent delay was defined by an algebraic relation, and in a suitable phase space it was possible to eliminate this auxiliary equation. Arino, Hadeler, Hbid, Magal [2, 25] and Hu, Wu [18] considered an equation where the auxiliary equation for the delay was given by an ordinary differential equation. Here we study a differential equation with a state-dependent delay where the delay is defined by two auxiliary equations: an algebraic equation and a differential equation with a discontinuous right hand side. The considered system is interesting from the theoretical point of view since previous results do not seem to work here. On the other hand, the system is a prototype of rate control problems with delays appearing naturally in queueing processes.

The particular model, that motivated our study, was introduced by Ranjan, La and Abed in [31, 30]. The problem is specified for a simple computer network, however, analogous models appear, e.g., in more general computer networks, in road networks, in biological networks, or in general in those processes where a bottleneck phenomenon slows down the performance or capacity of a system, see, e.g., [32, 10, 15]. The model is a fluid model of a network containing a single user and a single server. The user sends data by rate $x(t)$ to the server for procession. The user's transmission rate satisfies the bound $0 < a \leq x(t) \leq b$, where b is a user-specific physical limitation, and the lower bound a

is due to the fact that the user needs to probe the congestion level of the network by continuously transmitting data. The server processes the incoming data by the capacity $c \in (a, b)$. Kelly [19] introduced the utility $U(x)$ and the price $p(x)$ per unit flow of the procession, when the rate is x. Under natural conditions on the functions $U(\cdot)$ and $p(\cdot)$, there is an optimal rate $x_* \in (a, c)$ (balancing between the utility and the price of procession) as the unique maximum of the expression $U(x) - \int_0^x p(y)\,dy$ subject to the constraint $0 < x \le c$, see Kelly et al. [20]. In addition, [20] proposed an end user rate control algorithm as the differential equation

$$\dot{x}(t) = \kappa\big[x(t)U'(x(t)) - x(t)p(x(t))\big] \tag{3.1.1}$$

where $xU'(x)$ is the price per unit time the user is willing to pay for the procession, $xp(x)$ is the price charged by the server for procession, $\kappa > 0$ is a gain parameter. The solutions of equation (3.1.1) monotonically converge to x_* as $t \to \infty$. On the other hand, nonmonotone convergence and nonconvergent oscillation around x_* arise in some rate control problems. Equation (3.1.1) neglects the feedback delays appearing naturally in the system.

The rate control model of Ranjan, La and Abed [31, 30] takes the feedback delays into account. As the rate $x(t)$ can be larger than the capacity of the server, the data arriving at the server may form a single waiting line (a queue) before procession. Let $y(t)$ denote the length of the queue at time t. Suppose that it is bounded from above by $q > 0$, and the units of data reaching the queue with length q are lost. Then, assuming that the transmission time from the user to the server is $r_0 \ge 0$, it is natural that for the length $y(t)$ of the queue the differential equation

$$\dot{y}(t) = \begin{cases} x(t - r_0) - c & \text{if } 0 < y(t) < q, \\ [x(t - r_0) - c]^+ & \text{if } y(t) = 0, \\ -[x(t - r_0) - c]^- & \text{if } y(t) = q \end{cases} \tag{3.1.2}$$

is satisfied. Here, equation (3.1.2) is required to hold almost everywhere, and $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$ denote the positive and negative parts of u, respectively.

Suppose that a unit of data, whose procession was completed and the user received an acknowledgement about it at time t, arrived at the queue $\tau(t)$ time earlier, i.e., at time $t - \tau(t)$, and found a queue with length $y(t - \tau(t))$. As the capacity of the server is c, the given unit of data spent waiting time $z(t) = (1/c)y(t - \tau(t))$ in the queue before its procession started. Let r_1 denote the sum of the procession time and the transmission time from the server to the user. Then $\tau(t) = z(t) + r_1$, and this gives the algebraic equation

$$z(t) = \frac{1}{c} y(t - z(t) - r_1) \tag{3.1.3}$$

between y and z.

With the waiting time $z(t)$ and the transmission delays r_0, r_1, the user at time t receives an acknowledgement from the server about the procession of that unit of data which was sent at time $t - r_0 - z(t) - r_1$. The server determines a price for a unit rate when it arrives at the server, i.e., at time $t - z(t) - r_1$. When the procession of a unit ends, the server sends a signal to the user including the identification of the processed unit and the price information $p(x(t - z(t) - r_1))$. Then the user is able to estimate the price for the rate of data sent at $t - r_0 - z(t) - r_1$ as $x(t - r_0 - z(t) - r_1)p(x(t - z(t) - r_1))$. This led Ranjan, La and Abed [31, 30] to the rate control equation

$$\dot{x}(t) = \kappa\big[x(t)U'(x(t)) - x(t - r_0 - z(t) - r_1)p(x(t - z(t) - r_1))\big] \tag{3.1.4}$$

with a gain parameter $\kappa > 0$. See Figure 3.1. For similar models we refer to [1, 6].

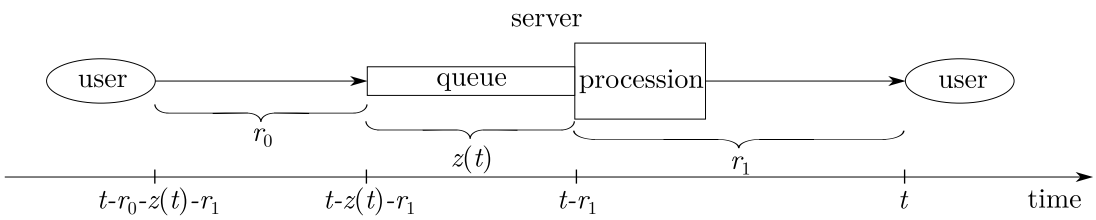


Figure 3.1: The process in time.

In this chapter we consider rate control equations (like (3.1.4)) with delay, where the delay is determined by two auxiliary equations, by (3.1.3) and (3.1.2), or only by (3.1.2). The primary aim of this chapter is to find a suitable framework to study the above types of rate control systems. We define a phase space where the corresponding initial value problem has a unique maximal solution. The solutions define a continuous semiflow, and the solution operators are Lipschitz continuous. We believe that this approach can be extended to handle a wide class of systems modeling networks with queueing delays. Observe that neither the classical results for equations with constant delay [12, 16] nor the recently developed results for equations with state-dependent delay [17, 35] do not work here.

The secondary aim is to apply the developed framework, and to show that the rate control defined by system (3.1.4), (3.1.2), (3.1.3) may lead to a slowly oscillating periodic rate around the optimal rate x_*, provided that the stationary solution $x = x_*$, $y = 0$, $z = 0$ is unstable and $r_0 = 0$. This answers affirmatively a conjecture of Ranjan and his coauthors [29, 28].

We give an overview on the main steps toward the results.

Set $r = r_0 + r_1 + q/c > 0$ as an upper bound for the delays. For a Lipschitz continuous

$\varphi : I \to \mathbb{R}$, let

$$\mathrm{lip}(\varphi) = \sup_{s\in I,\, t\in I,\, s<t} \left|\frac{\varphi(t)-\varphi(s)}{t-s}\right| \in [0,\infty) \quad \text{and}$$
$$\mathrm{slope}(\varphi) = \left\{\frac{\varphi(t)-\varphi(s)}{t-s} \,:\, s\in I,\ t\in I,\ s\neq t\right\} \subseteq \mathbb{R}.$$

First we consider a slightly more general system than (3.1.4), (3.1.2), (3.1.3), that is, in the equation we allow more general dependence on the length of the queue than that of (3.1.4), (3.1.2), (3.1.3), and equation (3.1.3) may or may not hold. Consider the equation

$$\dot{x}(t) = F(x_t, y_t) \tag{3.1.5}$$

together with (3.1.2) in the phase space $X \times Y$ where X, Y and F are defined as follows. An upper bound $K > 0$ for the absolute value of the right hand side of equation (3.1.5) comes from the nature of the problem. Then, by $x(t) \in [a,b]$ and the bound K, the subset

$$X = \left\{\varphi \in C_{[-r,0]} \;\middle|\; \varphi([-r,0]) \subseteq [a,b],\ \mathrm{lip}(\varphi) \le K\right\}$$

of $C_{[-r,0]}$ will contain all possible segments x_t. Analogously, by $y(t) \in [0,q]$, $x(t) \in [a,b]$ and equation (3.1.2), for the segments y_t, it is natural to introduce the subset

$$Y = \left\{\psi \in C_{[-r,0]} \;\middle|\; \varphi([-r,0]) \subseteq [0,q],\ \mathrm{slope}(\psi) \subseteq [a-c, b-c]\right\}$$

of $C_{[-r,0]}$. On $X \subset C_{[-r,0]}$, $Y \subset C_{[-r,0]}$, $X \times Y \subset C_{[-r,0]} \times C_{[-r,0]}$ we use the induced subspace topologies and the corresponding norms. By the Arzelà–Ascoli theorem, X, Y and $X \times Y$ are compact subsets of $C_{[-r,0]}$ and $C_{[-r,0]} \times C_{[-r,0]}$, respectively. Assume that the map $F : X \times Y \to \mathbb{R}$ has the following properties:

(H1) there exists $L > 0$ such that, for all $\varphi^1, \varphi^2 \in X$, $\psi^1, \psi^2 \in Y$,

$$\left|F(\varphi^1,\psi^1) - F(\varphi^2,\psi^2)\right| \le L\left(\left\|\varphi^1-\varphi^2\right\|_{[-r,0]} + \left\|\psi^1-\psi^2\right\|_{[-r,0]}\right);$$

(H2) $\max_{(\varphi,\psi)\in X\times Y} |F(\varphi,\psi)| \le K$;

(H3) there exists $r_2 \in (0, r_1]$ such that $F(\varphi,\psi^1) = F(\varphi,\psi^2)$ provided $\varphi \in X$, $\psi^1 \in Y$, $\psi^2 \in Y$, and $\psi^1|_{[-r,-r_2]} = \psi^2|_{[-r,-r_2]}$;

(H4) $F(\varphi,\psi) > 0$ if $\varphi \in X$, $\psi \in Y$, $\varphi(0) = a$, and $F(\varphi,\psi) < 0$ if $\varphi \in X$, $\psi \in Y$ and $\varphi(0) = b$.

A solution of system (3.1.5), (3.1.2) *in the phase space* $X \times Y$ on $[-r,\omega)$, $\omega \le \infty$, with initial condition $x_0 = \varphi \in X$, $y_0 = \psi \in Y$ is a pair of functions

$$x = x^{\varphi,\psi} : [-r,\omega) \to \mathbb{R} \quad \text{and} \quad y = y^{\varphi,\psi} : [-r,\omega) \to \mathbb{R}$$

such that

(i) $x_t \in X$ for all $t \in [0, \omega)$, $x_0 = \varphi$;

(ii) x is differentiable on $(0, \omega)$;

(iii) $y_t \in Y$ for all $t \in [0, \omega)$, $y_0 = \psi$;

(iv) equation (3.1.5) holds on $(0, \omega)$;

(v) equation (3.1.2) holds almost everywhere in $(0, \omega)$.

The solution $(x, y) = (x^{\varphi,\psi}, y^{\varphi,\psi})$ on $[-r, \omega)$ is called maximal if any other solution $(\widehat{x}, \widehat{y})$ with $\widehat{x}_0 = \varphi$, $\widehat{y}_0 = \psi$ is a restriction of (x, y).

In Section 3.3 we show that, under Hypotheses (H1)–(H4), for each $(\varphi, \psi) \in X \times Y$, system (3.1.5), (3.1.2) has a unique maximal solution $\left(x^{\varphi,\psi}, y^{\varphi,\psi}\right) : [-r, \infty) \to \mathbb{R}^2$. The solutions define the continuous semiflow

$$\Phi : [0, \infty) \times X \times Y \ni (t, \varphi, \psi) \mapsto \left(x_t^{\varphi,\psi}, y_t^{\varphi,\psi}\right) \in X \times Y,$$

and, for each $t \geq 0$, the solution operators $\Phi(t, \cdot, \cdot) : X \times Y \to X \times Y$ are Lipschitz continuous (Theorem 3.3.5). In order to sketch the main steps of the proof, let $(\varphi, \psi) \in X \times Y$ be given. As, by (H3), the value of $F(\varphi, \psi)$ does not depend on $\psi\big|_{[-r_2,0]}$, a standard contraction argument yields $T \in (0, r_2]$ and a unique $x : [-r, T] \to \mathbb{R}$ so that equation (3.1.5) holds on $(0, T)$, for arbitrary extension of $y_0 = \psi$ to $y : [-r, T] \to \mathbb{R}$. Next we redefine $y : [-r, T] \to \mathbb{R}$ on $(0, T]$ such that $y_t \in Y$ for all $t \in [0, T]$, and equation (3.1.2) holds almost everywhere on $[0, T]$ with $x : [-r, T] \to \mathbb{R}$ obtained in the first step. In order to appropriately redefine $y : [-r, T] \to \mathbb{R}$ on $[0, T]$, we extend the right hand side of (3.1.2) to an upper semicontinuous multivalued map, and apply a standard result from [11] for differential inclusions. These two steps combined give a unique solution $(x^{\varphi,\psi}, y^{\varphi,\psi})$ on $[-r, T]$. By the method of steps the solution can be uniquely extended to a maximal solution on some $[-r, \omega)$. Global existence, i.e., $\omega = \infty$, follows from (H4).

In order to see that system (3.1.4), (3.1.2), (3.1.3) is a particular case of system (3.1.5), (3.1.2) introduce $Z = [0, q/c] \subset \mathbb{R}$ as a state space for the variable $z(t)$. A cruical fact is the existence of a unique Lipschitz continuous map (Proposition 3.3.6) $\sigma : Y \to Z$ such that

$$\sigma(\psi) = \frac{1}{c}\psi(-\sigma(\psi) - r_1) \qquad (\psi \in Y).$$

Then, for a solution $(x, y) : [-r, \infty) \to \mathbb{R}^2$ of system (3.1.5), (3.1.2) in the phase space $X \times Y$, defining $z(t) = \sigma(y_t)$, $t \geq 0$, equation (3.1.3) is always satisfied for all $t \geq 0$.

Assume that a map $G : X \times Z \to \mathbb{R}$ is given such that, with the particular choice

$$F : X \times Y \ni (\varphi, \psi) \mapsto G(\varphi, \sigma(\psi)) \in \mathbb{R},$$

Hypotheses (H1)–(H4) hold. In this case system (3.1.5), (3.1.2) is equivalent to the system composed of the equations

$$\dot{x}(t) = G(x_t, z(t)), \tag{3.1.6}$$

(3.1.2) and (3.1.3). Then, in the phase space $X \times Y$, for each $(\varphi, \psi) \in X \times Y$, system (3.1.6), (3.1.2), (3.1.3) has the unique solution $x^{\varphi,\psi}[-r, \infty) \to \mathbb{R}$, $y^{\varphi,\psi} : [-r, \infty) \to \mathbb{R}$, $z^{\varphi,\psi} : [0, \infty) \to \mathbb{R}$ where $\left(x^{\varphi,\psi}, y^{\varphi,\psi}\right)$ is the solution of system (3.1.5), (3.1.2) with the above choice of F, and $z^{\varphi,\psi}(t) = \sigma(y_t^{\varphi,\psi})$, $t \geq 0$.

Defining the map G as

$$X \times Z \ni (\varphi, \zeta) \mapsto \kappa\big[\varphi(0)U'(\varphi(0)) - \varphi\left(-\zeta - r_0 - r_1\right) p\left(\varphi\left(-\zeta - r_1\right)\right)\big] \in \mathbb{R},$$

system (3.1.4), (3.1.2), (3.1.3) will be a particular case of system (3.1.6), (3.1.2), (3.1.3), see Section 3.5.

In Section 3.3 we show that system (3.1.6), (3.1.2), (3.1.3) can be studied not only in the phase space $X \times Y$, but also in $X \times Z$ with a different notion of solution. For given $(\varphi, \zeta) \in X \times Z$, the pair of functions $x : [-r, \infty) \to \mathbb{R}$, $z : [0, \infty) \to \mathbb{R}$ is called a *solution of system* (3.1.6), (3.1.2), (3.1.3) *in the phase space* $X \times Z$ if $x_t \in X$ and $z(t) \in Z$ for all $t \geq 0$, $x_0 = \varphi$, $z(0) = \zeta$, x is differentiable, equation (3.1.6) holds on $(0, \infty)$, moreover, there exists a function $y : [-r, \infty) \to \mathbb{R}$ with $y_t \in Y$, $z(t) = \sigma(y_t)$ for all $t \geq 0$, and equation (3.1.2) is satisfied almost everywhere on $[-\zeta - r_1, \infty)$.

The key technical result (see Section 3.3) to show that system (3.1.6), (3.1.2), (3.1.3) is well posed in $X \times Z$ is that there is a unique Lipschitz continuous map $\gamma : X \times Z \to Y$ so that $\psi = \gamma(\varphi, \zeta)$ satisfies $\psi(s) = c\zeta$ for $s \in [-r, -\zeta - r_1]$, and equation (3.1.2) holds a.e. in $[-\zeta - r_1, 0]$. In particular, $\zeta = (1/c)\psi(-\zeta - r_1)$. This means that the past of the length of the queue (that is $\psi \in Y$) can be recovered from the past of the rate (that is $\varphi \in X$) and from the present waiting time (that is $\zeta \in Z$). The maps

$$h : X \times Z \ni (\varphi, \zeta) \mapsto (\varphi, \gamma(\varphi, \zeta)) \in X \times Y, \;\; k : X \times Y \ni (\varphi, \psi) \mapsto (\varphi, \sigma(\psi)) \in X \times Z$$

are Lipschitz continuous, h is injective, and $k \circ h = \mathrm{id}_{X \times Z}$, $h \circ k\big|_{h(X \times Z)} = \mathrm{id}_{h(X \times Z)}$. Then (see Theorem 3.3.11), for each $(\varphi, \zeta) \in X \times Z$, there exists a unique solution $x^{\varphi,\zeta} : [-r, \infty) \to \mathbb{R}$, $z^{\varphi,\zeta} : [0, \infty) \to \mathbb{R}$ of system (3.1.6), (3.1.2), (3.1.3) in the phase space $X \times Z$ satisfying the initial condition $x_0^{\varphi,\zeta} = \varphi$, $z^{\varphi,\zeta}(0) = \zeta$. Moreover,

$$\Psi : [0, \infty) \times X \times Z \ni (t, \varphi, \zeta) \mapsto \left(x_t^{\varphi,\zeta}, z^{\varphi,\zeta}(t)\right) \in X \times Z$$

is a continuous semiflow on $X \times Z$, and $\Psi(t, \varphi, \zeta) = k(\Phi(t, h(\varphi, \zeta)))$ for all $t \geq 0$.

In Section 3.4 we assume $r_0 = 0$, $r_1 = 1$ and consider system (3.1.4), (3.1.2), (3.1.3). Condition $r_0 = 0$ guarantees a single delay in equation (3.1.4), $r_1 = 1$ can be achieved by rescaling the time. Then for the new variable $v = x - x_*$, by using $U'(x_*) = p(x_*)$, the rate control system (3.1.4), (3.1.2), (3.1.3) can be written as

$$\dot{v}(t) = -f(v(t)) - g(v(t - z(t) - 1)) \tag{3.1.7}$$

$$\dot{y}(t) = \begin{cases} v(t) - d & \text{if } 0 < y(t) < q \\ [v(t) - d]^+ & \text{if } y(t) = 0 \\ -[v(t) - d]^- & \text{if } y(t) = q \end{cases} \tag{3.1.8}$$

$$z(t) = \frac{1}{c} y(t - z(t) - 1) \tag{3.1.9}$$

where $f(v) = -\kappa\big[(v + x_*)U'(v + x_*) - x_*U'(x_*)\big]$, $g(v) = \kappa\big[(v + x_*)p(v + x_*) - x_*p(x_*)\big]$, and $d = c - x_* > 0$. With $A = a - x_* < 0$, $B = b - x_* > 0$, the nonlinearities f, g are assumed to be in $C^1([A, B], \mathbb{R})$ satisfying $0 \le f(\xi)/\xi \le f_1$, $0 < g(\xi)/\xi \le g_1$ for all $\xi \in [A, B] \setminus \{0\}$ for some $f_1 \ge 0$, $g_1 > 0$. Setting

$$K_0 = (f_1 + g_1)\max\{-A, B\}, \quad r = 1 + q/c, \quad K_1 = rK_0,$$

Theorem 3.3.11 implies that system (3.1.7), (3.1.8), (3.1.9) is well posed in the phase space $\mathcal{X} \times Z$ with

$$\mathcal{X} = \left\{\varphi \in C_{[-r,0]} \;\middle|\; \varphi([-r, 0]) \subseteq [A, B],\ \mathrm{lip}(\varphi) \le K_1\right\}.$$

A solution (v, z) of system (3.1.7), (3.1.8), (3.1.9) is called *slowly oscillatory* if for any two zeros $t_1,\ t_2$ of v with $t_1 < t_2$ the inequality $z(t_2) + 1 < t_2 - t_1$ holds. This means that the distance between consecutive zeros of v is larger than the delay.

Inspired by [26] and [25], introduce

$$W = \left\{(\varphi, \zeta) \in \mathcal{X} \times Z \;\middle|\; \varphi\big|_{[-r,-\zeta-1]} \equiv 0,\ s \mapsto \varphi(s)e^{f_1 s} \text{ is nondecreasing},\ \varphi(0) > 0\right\}$$

and $W_0 = W \cup \{(0, 0)\}$. Then, for each $(\varphi, \zeta) \in W$, the solution $v = v^{\varphi,\zeta} : [-r, \infty) \to \mathbb{R}$, $z = z^{\varphi,\zeta} : [0, \infty) \to \mathbb{R}$ is slowly oscillatory with infinite number of zeros. The second zero t_2 of v in $(0, \infty)$ determines $t_2^* > t_2$ so that $t_2 = t_2^* - z(t_2^*) - 1$, and a return map $P : W_0 \to W_0$ can be defined by $P(0, 0) = (0, 0)$ and

$$P(\varphi, \zeta) = \left(\widehat{v_{t_2^*}}, z(t_2^*)\right)$$

for $(\varphi, \zeta) \in W$ where $\widehat{v_{t_2^*}} \in \mathcal{X}$ is given by $\widehat{v_{t_2^*}}(s) = v(t_2^* + s)$ for $s \in [t_2 - t_2^*, 0]$, and $\widehat{v_{t_2^*}}(s) = 0$ for $s \in [-r, t_2 - t_2^*]$. A nontrivial fixed point of P corresponds to a slowly oscillating periodic solution. A classical tool, that we apply here as well, is Browder's non-ejective fixed point theorem. A large part of Section 3.4 is devoted to the construction of a suitable subset of $\mathcal{X} \times Z$ where Browder's theorem is applicable. We remark that, although the papers [26, 27, 2, 25, 39, 41, 38, 35] consider similar approach to get slowly oscillating periodic solutions, none of them can be directly applied here, because of the particular definition of the state-dependent queueing delay. Some steps of the proof are analogous, and other parts require new ideas.

It is a crucial result that $P(\varphi,\zeta)$ cannot decay too fast: there are constants $\theta > 0$, $\rho > 0$ with $v^{\varphi,\zeta}(t_2^*) \geq \theta\,(\varphi(0))^\rho$ for all $(\varphi,\zeta) \in W$. This fact allows to construct a C^2-function α on $[0, q/c]$ such that $\alpha(0) = 0$, $\alpha' > 0$, $\alpha'' > 0$ on $(0, q/c]$, $\alpha(q/c)$ is small enough, and the delayed inequality

$$\alpha(\xi - d/c) \geq \theta\,(\alpha(\xi))^\rho \qquad \left(\xi \in \left[\frac{d}{c}, \frac{q}{c}\right]\right)$$

holds. Defining the compact subsets

$$\begin{aligned} W_{\alpha,K_1} &= \left\{(\varphi,\zeta) \in W_0 \middle|\, \varphi(0) \geq \alpha(\zeta)\right\}, \\ W_{\alpha,K_0} &= \left\{(\varphi,\zeta) \in W_{\alpha,K_1} \middle|\, \mathrm{lip}(\varphi \leq K_0)\right\} \end{aligned} \tag{3.1.10}$$

of $\mathcal{X} \times Z$, the inclusion $P(W_{\alpha,K_1}) \subseteq W_{\alpha,K_0}$ is satisfied. However, W_{α,K_1} and W_{α,K_0} are not convex. Following [25], the subset

$$\begin{aligned} V_{\alpha,K_1} = \Big\{(\psi,\zeta) \in C_{[-1,0]} \times Z \,\Big|\, \psi([-1,0]) \subseteq [0,B],\ \mathrm{lip}(\psi) \leq K_1, \\ [-1,0] \ni s \mapsto \psi(s)e^{f_1 r s} \in \mathbb{R} \text{ is nondecreasing},\ \psi(-1) = 0,\ \psi(0) \geq \alpha(\zeta)\Big\} \end{aligned} \tag{3.1.11}$$

of $C_{[-1,0]} \times \mathbb{R}$ is compact and convex. Set V_{α,K_1} can be mapped into W_{α,K_1} by the streching map Q given by $Q(\psi,\zeta) = (\varphi,\zeta)$ with $\varphi(s) = \psi(s/(\zeta+1))$, $s \in [-\zeta-1, 0]$, and $\varphi\big|_{[-r,-\zeta-1]} \equiv 0$. The squeezing map R, defined by $R(\varphi,\zeta) = (\psi,\zeta)$ with $\psi(s) = \varphi((\zeta+1)s)$, $s \in [-1,0]$, maps W_{α,K_0} into V_{α,K_1}. Browder's theorem can be applied for the map

$$\Pi : V_{\alpha,K_1} \in (\psi,\zeta) \mapsto R \circ P \circ Q(\psi,\zeta) \in V_{\alpha,K_1}$$

to find a non-ejective fixed point of Π in V_{α,K_1}. This yields a non-ejective fixed point of P in W_{α,K_1} as well. The non-ejective fixed point is nontrivial provided $(0,0) \in W_{\alpha,K_1}$ is ejective. Ejectivity of $(0,0) \in W_{\alpha,K_1}$ follows in a standard way from that of the zero solution of the constant delay equation $\dot{v}(t) = -f(v(t)) - g(v(t-1))$.

Finally, Section 3.5 gives examples.

3.2 Preliminary results

In order to study the queue equation (3.1.2) we recall a basic result of [11] for differential inclusions.

Let $J = [t_0, t_1] \subset \mathbb{R}$ for some fixed $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$. Let $D \subseteq \mathbb{R}^j$ be closed. The multivalued map

$$F :\ J \times D \ \to\ 2^{\mathbb{R}^j} \setminus \{\emptyset\} = \left\{A \subseteq \mathbb{R}^j,\ A \neq \emptyset\right\}$$

is called *upper semicontinuous* if $F^{-1}(A)$ is closed in $J \times D$ whenever $A \subseteq \mathbb{R}^j$ is closed.

Let $\rho(y, D) = \inf_{z \in D} |y - z|$ for $y \in \mathbb{R}^j$. For $y \in D$ define

$$T_D(y) = \left\{ z \in \mathbb{R}^j : \liminf_{\lambda \to 0^+} \frac{1}{\lambda} \rho(y + \lambda z, D) = 0 \right\}.$$

The following existence result is Theorem 5.1 in [11]:

Theorem B. *Suppose that the multivalued map $F : J \times D \to 2^{\mathbb{R}^j} \setminus \{\emptyset\}$ is upper semicontinuous, for all $(t, y) \in J \times D$ the set $F(t, y)$ is closed and convex in $\mathbb{R}^j$,*

$$F(t, y) \cap T_D(y) \neq \emptyset \quad \textit{for all } (t, y) \in J \times D,$$

moreover, there is a Lebesgue integrable $c : J \to [0, \infty)$ such that, for all $(t, y) \in J \times D$,

$$\sup\{|z| : \ z \in F(t, y)\} \leq c(t)(1 + |y|)$$

holds. Then, for each $y_0 \in D$, there exists an absolutely continuous $y : J \to D$ such that $y(t_0) = y_0$ and the inclusion

$$\dot{y}(t) \in F(t, y(t)) \quad \textit{holds a.e. on } [t_0, t_1].$$

Assume that $\mathcal{E}$ is a Banach space, $\mathcal{C} \subset \mathcal{E}$ is compact and convex in $\mathcal{E}$, the map $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is continuous. A fixed point $x_0 \in \mathcal{C}$ of $\mathcal{F}$ is said to be *ejective* if there exists an open neighborhood $\mathcal{U}$ of x_0 in $\mathcal{C}$ such that for each $x \in \mathcal{U} \setminus \{x_0\}$ there exists a positive integer $k(x)$ such that for the iterate $\mathcal{F}^{k(x)}(x) \in \mathcal{C} \setminus \mathcal{U}$ holds. In Section 3.4 we will apply the following result [7] of Browder on the existence of a non-ejective fixed point.

Theorem C. *Assume that $\mathcal{E}$ is a Banach space, $\mathcal{C} \subset \mathcal{E}$ is an infinite dimensional compact and convex subset of $\mathcal{E}$, the map $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is continuous. Then $\mathcal{F}$ has a non-ejective fixed point.*

For the application of the above result, we have to guarantee the ejectivity of the trivial fixed point of a return map. The proof of the ejectivity uses properties of the linear autonomous equation with constant delay

$$\dot{w}(t) = -\mu w(t) - \nu w(t - 1) \tag{3.2.1}$$

where $\mu \geq 0$ and $\nu > 0$. We recall some basic results from [12, 16, 40]. It is well known that every $\varphi \in C_{[-1,0]}$ uniquely determines a solution $w^\varphi : [-1, \infty) \to \mathbb{R}$ of equation (3.2.1) with $w^\varphi|_{[-1,0]} = \varphi$, and the solutions define the strongly continuous semigroup $(T(t))_{t \geq 0}$ on $[0, \infty) \times C_{[-1,0]}$. The spectrum of the generator consists of the solutions $\lambda \in \mathbb{C}$ of the characteristic equation $\lambda + \mu + \nu e^{-\lambda} = 0$. Assume $\nu > e^{-\mu-1}$. Then all points in the spectrum form a sequence of complex conjugate pairs $(\lambda_j, \overline{\lambda_j})_{j=1}^{\infty}$ with $\operatorname{Re} \lambda_j >$

$\operatorname{Re}\lambda_{j+1}, \operatorname{Im}\lambda_j \in ((2j-2)\pi, (2j-1)\pi)$ for all $j \in \mathbb{N}$, and $\operatorname{Re}\lambda_j \to -\infty$ as $j \to \infty$. An explicit criterion for $\operatorname{Re}\lambda_1 > 0$ is

$$\nu > \frac{\vartheta}{\sin\vartheta} \text{ where } \vartheta \in (0,\pi) \text{ is the unique solution of } \mu = -\vartheta\cot\vartheta. \tag{3.2.2}$$

Let $\mathcal{L}$ and $\mathcal{Q}$ denote the realified generalized eigenspaces of the generator associated with the spectral sets $\{\lambda_1, \overline{\lambda_1}\}$ and $\{\lambda_k, \overline{\lambda_k} : k \geq 2\}$, respectively. Then $C_{[-1,0]} = \mathcal{L} \oplus \mathcal{Q}$. A basis of $\mathcal{L}$ is given by the restrictions of the functions

$$t \mapsto e^{\operatorname{Re}\lambda_1 t}\sin(\operatorname{Im}\lambda_1 t), \qquad t \mapsto e^{\operatorname{Re}\lambda_1 t}\cos(\operatorname{Im}\lambda_1 t)$$

to the interval $[-1,0]$.

Let $S \subset C_{[-1,0]} \setminus \{0\}$ be the set of functions with at most one sign change in $[-1,0]$. The set S is invariant, i.e., $T(t)S \subseteq S$ for all $t \geq 0$. Moreover,

$$S \cap \mathcal{Q} = \emptyset.$$

Proposition 3.2.1. *If $\mu \geq 0$ and $\nu > 0$ are given such that $\operatorname{Re}\lambda_1 > 0$, i.e.,* (3.2.2) *holds, and $\varphi \in S$, then the solution w^φ of equation* (3.2.1) *is unbounded on* $[-1,\infty)$.

Proof. Let $\varphi \in S$ and $w = w^\varphi$. From $C_{[-1,0]} = \mathcal{L} \oplus \mathcal{Q}$, $S \cap \mathcal{Q} = \emptyset$ and $\varphi \neq 0$ it follows that $\varphi = \varphi^{\mathcal{L}} + \varphi^{\mathcal{Q}}$ with $\varphi^{\mathcal{L}} \in \mathcal{L} \setminus \{0\}$, $\varphi^{\mathcal{Q}} \in \mathcal{Q}$. Then $w = w_{\mathcal{L}} + w_{\mathcal{Q}}$ where $w_{\mathcal{L}} = w^{\varphi^{\mathcal{L}}}$ and $w_{\mathcal{Q}} = w^{\varphi^{\mathcal{Q}}}$. As $\varphi^{\mathcal{L}} \in \mathcal{L} \setminus \{0\}$, there exist $k_1,\ k_2 \in \mathbb{R}$ with $k_1^2 + k_2^2 \neq 0$ so that, for all $t \geq -1$,

$$w_{\mathcal{L}}(t) = e^{\operatorname{Re}\lambda_1 t}\left[k_1\sin(\operatorname{Im}\lambda_1 t) + k_2\cos(\operatorname{Im}\lambda_1 t)\right].$$

The estimate on the complementary space $\mathcal{Q}$ (see, e.g., [12] or [16]) implies that there are $\delta > 0$ and $M > 0$ such that, for all $t \geq -1$,

$$|w_{\mathcal{Q}}(t)| \leq M e^{(\operatorname{Re}\lambda_1 - \delta)t}.$$

Then, by $\operatorname{Re}\lambda_1 > 0$, it easly follows that w is unbounded. □

3.3 The solution semiflow

Assume that $r_0, r_1, r_2, q, a, b, c, K, L$ are given constants as in Section 3.1, and Hypotheses (H1)–(H4) hold. First we consider system (3.1.5), (3.1.2). Condition (H3) means that $F(\varphi,\psi)$ does not depend on $\psi|_{[-r_2,0]}$. Consequently, for given $\varphi \in X$ and $\psi \in Y$, we can find $x : [-r,T] \to \mathbb{R}$ with $x_0 = \varphi$ satisfying equation (3.1.5) on an interval $[0,T]$ for some $T \in (0, r_2)$, no matter how $y|_{[-r,0]} = \psi$ is extended to $[-r,T]$. This is done in the next proposition by using a standard fixed point technique. After that $x : [-r,T] \to \mathbb{R}$ is obtained, we will be able to determine $y : [-r,T] \to \mathbb{R}$ satisfying equation (3.1.2) on $[0,T]$. These two results together give a solution of system (3.1.5), (3.1.2) on $[-r,T]$. Repeating this procedure by time-T steps a global solution will be obtained.

Proposition 3.3.1. *Let $T \in (0, r_2]$ be fixed such that $TL < 1$. For every $(\varphi, \psi) \in X \times Y$ there exists a unique function $x = x(\varphi, \psi) : [-r, T] \to \mathbb{R}$ such that $x_0 = \varphi$, $x_t \in X$ for all $t \in [0, T]$, x is differentiable on $(0, T]$, and, for each $y : [-r, T] \to \mathbb{R}$ with $y_0 = \psi$ and $y_t \in Y$ for all $t \in [0, T]$, x satisfies equation (3.1.5) on $(0, T]$. Moreover, the Lipschitz continuity property*

$$\left\| x\left(\varphi^1, \psi^1\right) - x\left(\varphi^2, \psi^2\right) \right\|_{[-r,T]} \leq \frac{\|\varphi^1 - \varphi^2\|_{[-r,0]} + TL\,\|\psi^1 - \psi^2\|_{[-r,0]}}{1 - TL}$$

holds for all (φ^1, ψ^1), (φ^2, ψ^2) in $X \times Y$.

Proof. Let $(\varphi, \psi) \in X \times Y$ be given. Define $\widehat{\varphi}, \widehat{\psi} \in C_{[-r,T]}$ by

$$\widehat{\varphi}(t) = \begin{cases} \varphi(t) & \text{if } t \leq 0, \\ \varphi(0) & \text{if } t > 0, \end{cases} \qquad \widehat{\psi}(t) = \begin{cases} \psi(t) & \text{if } t \leq 0, \\ \psi(0) & \text{if } t > 0. \end{cases}$$

The set

$$M = \left\{ u \in C_{[0,T]} : u(0) = 0,\ \operatorname{lip}(u) \leq K \right\},$$

is a complete metric space with distance $d(u, v) = \|u - v\|_{[0,T]}$. Introduce the map $m : M \times [a, b] \to C_{[-r,T]}$ by

$$m(u, \xi)(t) = \begin{cases} 0 & \text{if } t \in [-r, 0], \\ \min\{\max\{u(t), a - \xi\}, b - \xi\} & \text{if } t \in [0, T]. \end{cases}$$

Function $m(u, \xi)$ is a trivial extension of u to $[-r, 0]$, and it cuts the values of u on $[0, T]$ so that $m(u, \xi)(t) \in [a - \xi, b - \xi]$ is satisfied. Then it is clear that

$$\widehat{\varphi}_t + m_t(u, \varphi(0)) \in X, \quad \widehat{\psi}_t \in Y \text{ for all } t \in [0, T],$$

and $[0, T] \mapsto \widehat{\varphi}_t + m_t(u, \varphi(0)) \in X$, $[0, T] \mapsto \widehat{\psi}_t \in Y$ are continuous maps. It is easy to see that

$$\left\| m\left(u^1, \xi\right) - m\left(u^2, \xi\right) \right\|_{[-r,T]} \leq \left\| u^1 - u^2 \right\|_{[0,T]} \qquad (u^1 \in M,\ u^2 \in M,\ \xi \in [a, b]).$$

Define the map $\mathcal{N} : X \times Y \times M \to M$ as follows:

$$\mathcal{N}(\varphi, \psi, u)(t) = \int_0^t F\left(\widehat{\varphi}_s + m_s(u, \varphi(0)), \widehat{\psi}_s\right) ds, \qquad t \in [0, T].$$

By (H1) and (H2), F is continuous and $|F| \leq K$. Therefore, it is obvious that $\mathcal{N}(\varphi, \psi, u) \in M$.

Now, fix $(\varphi, \psi) \in X \times Y$. For functions $u^1, u^2 \in M$, by the definition of $\mathcal{N}$, m, and by (H1) and the Lipschitz property of m, we have

$$\begin{aligned} &\left\| \mathcal{N}\left(\varphi, \psi, u^1\right) - \mathcal{N}\left(\varphi, \psi, u^2\right) \right\|_{[0,T]} \\ &= \max_{t \in [0,T]} \left| \int_0^t \left[F\left(\widehat{\varphi}_s + m_s\left(u^1, \varphi(0)\right), \widehat{\psi}_s\right) - F\left(\widehat{\varphi}_s + m_s\left(u^2, \varphi(0)\right), \widehat{\psi}_s\right) \right] ds \right| \\ &\leq \int_0^T L \left\| m\left(u^1, \varphi(0)\right) - m\left(u^2, \varphi(0)\right) \right\|_{[-r,T]} ds \leq TL \left\| u^1 - u^2 \right\|_{[0,T]}. \end{aligned}$$

Since $TL < 1$, for all $(\varphi, \psi) \in X \times Y$, the map $M \ni u \mapsto \mathcal{N}(\varphi, \psi, u) \in M$ is a contraction. Therefore, as M is a complete metric space, there is a unique fixed point $u^*(\varphi, \psi) \in M$.

Let $(\varphi^i, \psi^i) \in X \times Y$ and $u_i^* = u^*(\varphi^i, \psi^i)$, $i = 1, 2$. From the obvious inequality

$$\left\|\widehat{\varphi}^1 + m\left(u, \varphi^1(0)\right) - \widehat{\varphi}^2 - m\left(u, \varphi^2(0)\right)\right\|_{[-r,T]} \leq \left\|\varphi^1 - \varphi^2\right\|_{[-r,0]},$$

it follows that

$$\begin{aligned}
&\|u_1^* - u_2^*\|_{[0,T]} = \left\|\mathcal{N}\left(\varphi^1, \psi^1, u_1^*\right) - \mathcal{N}\left(\varphi^2, \psi^2, u_2^*\right)\right\|_{[0,T]} \\
&\leq \max_{t\in[0,T]} \left| \int_0^t \left[F\left(\widehat{\varphi}_s^1 + m_s\left(u_1^*, \varphi^1(0)\right), \widehat{\psi}_s^1\right) - F\left(\widehat{\varphi}_s^2 + m_s\left(u_2^*, \varphi^2(0)\right), \widehat{\psi}_s^2\right)\right] ds \right| \\
&\leq \int_0^T L\Big(\left\|\widehat{\varphi}^1 + m\left(u_1^*, \varphi^1(0)\right) - \widehat{\varphi}^2 - m\left(u_1^*, \varphi^2(0)\right)\right\|_{[-r,T]} \\
&+ \left\|m\left(u_1^*, \varphi^2(0)\right) - m\left(u_2^*, \varphi^2(0)\right)\right\|_{[-r,T]} + \left\|\widehat{\psi}_1 - \widehat{\psi}_2\right\|_{[-r,T]} \Big) ds \\
&\leq TL\left(\left\|\varphi^1 - \varphi^2\right\|_{[-r,0]} + \|u_1^* - u_2^*\|_{[0,T]} + \left\|\psi^1 - \psi^2\right\|_{[-r,0]}\right).
\end{aligned}$$

Consequently,

$$\left\|u^*\left(\varphi^1, \psi^1\right) - u^*\left(\varphi^2, \psi^2\right)\right\|_{[0,T]} \leq \frac{TL}{1 - TL}\left(\left\|\varphi^1 - \varphi^2\right\|_{[-r,0]} + \|\psi_1 - \psi_2\|_{[-r,0]}\right).$$

We claim that, for all $(\varphi, \psi) \in X \times Y,\ t \in (0, T]$,

$$\varphi(0) + u^*(\varphi, \psi)(t) \in (a, b).$$

If $t_0 \in [0, T]$ and $\varphi(0) + u^*(\varphi, \psi)(t_0) = a$, we have $\widehat{\varphi}(t_0) + m(u^*(\varphi, \psi), \varphi(0))(t_0) = a$. Then by (H4), $F(\widehat{\varphi}_{t_0} + m_{t_0}(u^*(\varphi, \psi), \varphi(0)), \widehat{\psi}_{t_0}) > 0$. By continuity, it follows that there is a $\delta > 0$ so that $F(\widehat{\varphi}_t + m_t(u^*(\varphi, \psi), \varphi(0)), \widehat{\psi}_t) > 0$ for all $t \in (t_0 - \delta, t_0 + \delta) \cap [0, T]$. The fixed point equation for $u^*(\varphi, \psi)$ implies that $t \mapsto u^*(\varphi, \psi)(t)$ strictly increases in $(t_0 - \delta, t_0 + \delta) \cap [0, T]$. Hence it is easy to see that

$$\varphi(0) + u^*(\varphi, \psi)(t) > a \text{ for all } t \in (0, T].$$

Analogously, $\varphi(0) + u^*(\varphi, \psi)(t) < b$ holds for all $t \in (0, T]$. So the claim is true.

A consequence of the claim is that

$$m(u^*(\varphi, \psi), \varphi(0))(t) = u^*(\varphi, \psi)(t) \text{ for all } t \in [0, T],$$

and the function

$$x(t) = x(\varphi, \psi)(t) = \begin{cases} \varphi(t) & \text{if } t \in [-r, 0], \\ \varphi(0) + u^*(\varphi, \psi)(t) & \text{if } t \in [0, T] \end{cases}$$

satisfies $x_0 = \varphi$, $x_t \in X$ for $t \in [0, T]$, x is differentiable on $(0, T]$, and equation (3.1.5) holds on $(0, T]$ with the particular choice $y = \widehat{\psi}$. Observe that, by Hypothesis (H3), the above construction gives the same $x(\varphi, \psi)$ for any $y : [-r, T] \to \mathbb{R}$ so that $y_0 = \psi$ and $y_t \in Y$ for $t \in [0, T]$.

Finally, it is straightforward to get the estimate

$$\begin{aligned}\left\|x\left(\varphi^1, \psi^1\right) - x\left(\varphi^2, \psi^2\right)\right\|_{[-r,T]} &\leq \left\|\varphi^1 - \varphi^2\right\|_{[-r,0]} + \left\|u^*\left(\varphi^1, \psi^1\right) - u^*\left(\varphi^2, \psi^2\right)\right\|_{[0,T]} \\ &\leq \frac{1}{1 - TL}\left\|\varphi^1 - \varphi^2\right\|_{[-r,0]} + \frac{TL}{1 - TL}\left\|\psi^1 - \psi^2\right\|_{[-r,0]}.\end{aligned}$$

This completes the proof. □

In the next step we study equation (3.1.2). Since we need the same type of result in another situation as well, a slightly more general version is considered.

Let $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$. Assume that a function $\xi \in C([t_0, t_1], [a, b])$ is given. Let $y^0 \in [0, q]$ be fixed. We consider the equation

$$\dot{y}(t) = \begin{cases} \xi(t) - c & \text{if } 0 < y(t) < q, \\ [\xi(t) - c]^+ & \text{if } y(t) = 0, \\ -[\xi(t) - c]^- & \text{if } y(t) = q \end{cases} \tag{3.3.1}$$

on the interval $[t_0, t_1]$ with initial condition $y(t_0) = y^0$.

Proposition 3.3.2. *For each $\xi \in C([t_0, t_1], [a, b])$ and each $y^0 \in [0, q]$ there exists a unique Lipschitz continuous function $y = y(\xi, y^0) : [t_0, t_1] \to [0, q]$ such that $y(t_0) = y^0$,* slope$(y) \subseteq [a - c, b - c]$*, and equation* (3.3.1) *holds almost everywhere in $[t_0, t_1]$. In addition, $y(\xi, y^0)$ is Lipschitz continuous in ξ, y^0, namely, for all $\xi^1, \xi^2 \in C([t_0, t_1], [a, b])$ and $y^{0,1}, y^{0,2} \in [0, q]$,*

$$\left\|y(\xi^1, y^{0,1}) - y(\xi^2, y^{0,2})\right\|_{[t_0,t_1]} \leq \left|y^{0,1} - y^{0,2}\right| + (t_1 - t_0)\left\|\xi^1 - \xi^2\right\|_{[t_0,t_1]}.$$

Proof. Let $\xi \in C([t_0, t_1], [a, b])$ and $y^0 \in [0, q]$ be fixed. Define the map $h : [t_0, t_1] \times [0, q] \to \mathbb{R}$ by

$$h(t, y) = \begin{cases} \xi(t) - c & \text{if } 0 < y < q, \\ [\xi(t) - c]^+ & \text{if } y = 0, \\ -[\xi(t) - c]^- & \text{if } y = q. \end{cases}$$

Then equation (3.3.1) with $y(t_0) = y^0$ on $[t_0, t_1]$ can be written as an initial value problem

$$\begin{cases} \dot{y}(t) = h(t, y(t)) & \text{a.e. for } t \in [t_0, t_1], \\ y(t_0) = y^0. \end{cases} \tag{3.3.2}$$

The remaining part of the proof is divided into three steps. In Steps 1–2 we show existence, in Step 3 uniqueness and the Lipschitz property are obtained.

Step 1. We extend h to a multivalued function $\widetilde{h} : [t_0, t_1] \times [0, q] \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ as follows:

$$\widetilde{h}(t, y) = \begin{cases} \{\xi(t) - c\} & \text{if } y \in (0, q), \\ & \text{or } y = 0 \text{ and } \xi(t) \geq c, \\ & \text{or } y = q \text{ and } \xi(t) \leq c, \\ [\xi(t) - c, 0] & \text{if } y = 0 \text{ and } \xi(t) < c, \\ [0, \xi(t) - c] & \text{if } y = q \text{ and } \xi(t) > c. \end{cases}$$

It is easy to check that $\widetilde{h}$ is an upper semicontinuous function. We apply Theorem B by choosing $j = 1$, $D = [0, q]$, $J = [t_0, t_1]$, $F = \widetilde{h}$. Clearly, $T_D(y) = \mathbb{R}$ for $y \in (0, q)$, $T_D(0) = [0, \infty)$ and $T_D(q) = (-\infty, 0]$. It is obvious that the conditions of Theorem B are satisfied with $c(t) = \max\{c - a, b - c\}$. Therefore, there is an absolutely continuous $y = y(\xi, y^0) : [t_0, t_1] \to [0, q]$ such that

$$\dot{y}(t) \in \widetilde{h}(t, y(t)) \quad \text{a.e. for } \ t \in [t_0, t_1] \tag{3.3.3}$$

and $y(t_0) = y^0$.

Step 2. We show that for the function $y = y(\xi, y^0)$, obtained in Step 1, equation (3.3.1) holds almost everywhere, and $y(t_0) = y^0$.

Assume that $t \in (t_0, t_1)$ is given such that $\dot{y}(t)$ exists and $\dot{y}(t) \in \widetilde{h}(t, y(t))$.

If $y(t) \in (0, q)$ then $\widetilde{h}(t, y(t)) = \{\xi(t) - c\}$, and consequently $\dot{y}(t) = h(t, y(t))$. If $y(t) = 0$ then necessarily $\dot{y}(t) = 0$. From $\dot{y}(t) = 0 \in \widetilde{h}(t, 0)$ it follows that $\xi(t) \leq c$, and thus $0 = \dot{y}(t) = [\xi(t) - c]^+ = h(t, y(t))$. The case $y(t) = q$ is analogous.

Therefore, $y = y(\xi, y^0)$ satisfies equation (3.3.2). Then, by the definition of $h(t, y)$ and $\xi([t_0, t_1]) \subseteq [a, b]$, it is clear that (3.3.1) holds almost everywhere for y, $y(t_0) = y^0$, and $\text{slope}(y) \subseteq [a - c, b - c]$.

Step 3. Let $\xi^1, \xi^2 \in C([t_0, t_1], \mathbb{R})$, $y^{0,1}, y^{0,2} \in [0, q]$, $y^1 = y(\xi^1, y^{0,1})$, $y^2 = y(\xi^2, y^{0,2})$. Then the map $[t_0, t_1] \ni t \mapsto |y^1(t) - y^2(t)| \in \mathbb{R}$ is absolutely continuous.

Claim 3.3.3. *For $\xi^1, \xi^2 \in C([t_0, t_1], \mathbb{R})$, $y^{0,1}, y^{0,2} \in [0, q]$, $y^1 = y(\xi^1, y^{0,1})$, $y^2 = y(\xi^2, y^{0,2})$*

$$\left|\dot{y}^1(s) - \dot{y}^2(s)\right| \leq \left|\xi^1(s) - \xi^2(s)\right| \qquad \textit{holds a.e. in } [t_0, t_1].$$

For almost all $s \in (t_0, t_1)$ the derivative $\dot{y}^i(s)$ exists with $\dot{y}^i(s) = h^i(s, y^i(s))$, where h^i is the map constructed as h above with ξ replaced by ξ^i, $i = 1, 2$. Fix such an $s \in (t_0, t_1)$. We distinguish 4 cases.

Case 1. $y^i(s) \in (0, q)$, $i \in \{1, 2\}$. Then, by the definition of h^1,h^2, $|\dot{y}^1(s) - \dot{y}^2(s)| = |\xi^1(s) - \xi^2(s)|$.

Case 2. $y^i(s) \in \{0, q\}$, $i \in \{1, 2\}$. In this case $\dot{y}^1(s) = \dot{y}^2(s) = 0$, and hence $|\dot{y}^1(s) - \dot{y}^2(s)| = 0 \leq |\xi^1(s) - \xi^2(s)|$.

Case 3. $y^1(s) = 0$, $y^2(s) \in (0, q)$. Then $\dot{y}^1(s) = 0$ and consequently $\xi^1(s) \leq c$. In addition,

$$\left|\dot{y}^1(s) - \dot{y}^2(s)\right| = \left|-\left(c - \xi^2(s)\right)\right| = \left|\xi^2(s) - c\right| \leq \left|\xi^2(s) - \xi^1(s)\right|.$$

Case 4. $y^1(s) \in (0, q)$, $y^2(s) = q$. Then $\dot{y}^2(s) = 0$ and $\xi^2(s) \geq c$ follows. Hence

$$\left|\dot{y}^1(s) - \dot{y}^2(s)\right| = \left|-\left(\xi^1(s) - c\right)\right| = \left|c - \xi^1(s)\right| \leq \left|\xi^2(s) - \xi^1(s)\right|.$$

The remaining cases can be obtained by changing the indices. This completes the proof of the claim.

Since

$$y^1(t) - y^2(t) = y^{0,1} - y^{0,2} + \int_{t_0}^{t} \frac{d}{ds}[y^1(s) - y^2(s)]\, ds \quad \text{for } t \in [t_0, t_1],$$

we have

$$\begin{aligned}\left\|y^1 - y^2\right\|_{[t_0,t_1]} &\leq \left|y^{0,1} - y^{0,2}\right| + \int_{t_0}^{t_1} \left|\dot{y}^1(s) - \dot{y}^2(s)\right| ds \\ &= \left|y^{0,1} - y^{0,2}\right| + (t_1 - t_0) \left\|\xi^1 - \xi^2\right\|_{[t_0,t_1]}.\end{aligned}$$

This implies the uniqueness of $y(\xi, y^0)$, and the Lipschitz continuity of $y(\xi, y^0)$ with respect to ξ and y^0. The proof is complete. □

The following corollary is immediate from Proposition 3.3.2.

Corollary 3.3.4. *Let $T > 0$. For all $\widetilde{\xi} \in C([-r, T], [a, b])$ and $\psi \in Y$ there exists a unique Lipschitz continuous function $y = y(\widetilde{\xi}, \psi) : [-r, T] \to [0, q]$ such that $y_0 = \psi$, $\operatorname{slope}(y) \subseteq [a-c, b-c]$, and equation (3.3.1) with $\xi(t) = \widetilde{\xi}(t-r_0)$ holds almost everywhere in $[0, T]$. In addition, $y(\widetilde{\xi}, \psi)$ is Lipschitz continuous in $\widetilde{\xi}, \psi$, namely, for all $\widetilde{\xi^1}, \widetilde{\xi^2} \in C([-r, T], [a, b])$ and $\psi^1, \psi^2 \in Y$,*

$$\left\|y(\widetilde{\xi^1}, \psi^1) - y(\widetilde{\xi^2}, \psi^2)\right\|_{[-r,T]} \leq \left\|\psi^1 - \psi^2\right\|_{[-r,0]} + T\left\|\widetilde{\xi^1} - \widetilde{\xi^2}\right\|_{[-r,T]}.$$

Now we are in a position to prove existence, uniqueness, and continuous dependence of the solutions of system (3.1.5), (3.1.2).

Theorem 3.3.5. *For each $(\varphi, \psi) \in X \times Y$ there exists a unique solution*

$$x^{\varphi,\psi} : [-r, \infty) \to \mathbb{R}, \qquad y^{\varphi,\psi} : [-r, \infty) \to \mathbb{R}$$

of system (3.1.5), (3.1.2) on $[-r, \infty)$ satisfying the initial condition $x_0^{\varphi,\psi} = \varphi$, $y_0^{\varphi,\psi} = \psi$. The mapping

$$\Phi : [0, \infty) \times X \times Y \ni (t, \varphi, \psi) \mapsto \left(x_t^{\varphi,\psi}, y_t^{\varphi,\psi}\right) \in X \times Y$$

defines a continuous semiflow on $X \times Y$. In addition, Φ has the following Lipschitz continuity property

$$\left\|\Phi\left(t, \varphi^1, \psi^1\right) - \Phi\left(t, \varphi^2, \psi^2\right)\right\|_{X\times Y} \leq \left\|\left(\varphi^1, \psi^1\right) - \left(\varphi^2, \psi^2\right)\right\|_{X\times Y} e^{t(1+L)}.$$

Proof. Let $T \in (0, r_2]$, $TL < 1$ and $(\varphi, \psi) \in X \times Y$. By Proposition 3.3.1 there exists a unique function $x = x(\varphi, \psi) : [-r, T] \to \mathbb{R}$ such that $x_0 = \varphi$, $x_t \in X$ for all $t \in [0, T]$, x is differentiable on $(0, T]$, x satisfies equation (3.1.5) on $(0, T]$, and the function $y : [-r, T] \to \mathbb{R}$ in (3.1.5) is arbitrary with $y_0 = \psi$ and $y_t \in Y$ for all $t \in [0, T]$. By Corollary 3.3.4, with $\widetilde{\xi} = x(\varphi, \psi)$, we can choose a unique $y = y(x(\varphi, \psi), \psi) : [-r, T] \to \mathbb{R}$ such that $y_0 = \psi$, $y_t \in Y$ for all $t \in [0, T]$, and equation (3.1.2) holds almost everywhere in $[0, T]$.

The functions $x^{\varphi,\psi} : [-r, \infty) \to \mathbb{R}$ and $y^{\varphi,\psi} : [0, \infty) \to \mathbb{R}$ are defined as follows. Set $x^{\varphi,\psi}(t) = x(\varphi, \psi)(t)$, $y^{\varphi,\psi}(t) = y(x(\varphi, \psi), \psi)(t)$ for $t \in [-r, T]$. Hence we can define $\widetilde{\varphi} = x_T^{\varphi,\psi} \in X$ and $\widetilde{\psi} = y_T^{\varphi,\psi} \in Y$. For $(\widetilde{\varphi}, \widetilde{\psi}) \in X \times Y$, the functions $x(\widetilde{\varphi}, \widetilde{\psi})$ and $y(\widetilde{\varphi}, \widetilde{\psi})$ can be constructed as above. Set $x^{\varphi,\psi}(t) = x(\widetilde{\varphi}, \widetilde{\psi})(t - T)$, $y^{\varphi,\psi}(t) = y(x(\widetilde{\varphi}, \widetilde{\psi}), \widetilde{\psi})(t - T)$ for $t \in [T, 2T]$. This procedure can be repeated to define $x^{\varphi,\psi}$ and $y^{\varphi,\psi}$ on the interval $[-r, \infty)$. The differentiability of $x^{\varphi,\psi}$ on $(0, \infty)$ follows from the continuity of the map $[0, \infty) \ni t \mapsto F(x_t^{\varphi,\psi}, y_t^{\varphi,\psi}) \in \mathbb{R}$. It is not difficult to see that $x^{\varphi,\psi}$ and $y^{\varphi,\psi}$ will be the unique solution of system (3.1.5), (3.1.2) on $[-r, \infty)$ with initial condition $x_0^{\varphi,\psi} = \varphi$, $y_0^{\varphi,\psi} = \psi$. Defining

$$\Phi(t, \varphi, \psi) = \left(x_t^{\varphi,\psi}, y_t^{\varphi,\psi}\right) \qquad (t \geq 0,\ \varphi \in X,\ \psi \in Y),$$

the uniqueness clearly guarantees the semigroup property. Continuity of $[0, \infty) \ni t \mapsto \Phi(t, \varphi, \psi) \in X \times Y$ is a consequence of the Lipschitz property of X and Y. Combining these properties with the stated Lipschitz continuity property of Φ it is easy to conclude the continuity of Φ.

So, it remains to prove that Φ is Lipschitz continuous in φ, ψ as claimed in the Theorem. Before the proof, we remark that the unique pair $x^{\varphi,\psi}$, $y^{\varphi,\psi}$ does not depend on T used in Proposition 3.3.1 and later. Hence, the construction works for all $T \in (0, r_2]$ with $TL < 1$.

Let $(\varphi^i, \psi^i) \in X \times Y$, $x^i = x^{\varphi^i,\psi^i}$, $y^i = y^{\varphi^i,\psi^i}$, $i = 1, 2$. For each $t \geq 0$ with $T \in (0, r_2]$ and $TL < 1$, by using Proposition 3.3.1 and Corollary 3.3.4, we have the following estimate:

$$\begin{aligned}
&\left\|x^1 - x^2\right\|_{[t-r,t+T]} + \left\|y^1 - y^2\right\|_{[t-r,t+T]} \\
&\leq \left\|x^1 - x^2\right\|_{[t-r,t+T]} + \left\|y^1 - y^2\right\|_{[t-r,t]} + T\left\|x^1 - x^2\right\|_{[t-r,t+T]} \\
&= (1 + T)\left\|x^1 - x^2\right\|_{[t-r,t+T]} + \left\|y^1 - y^2\right\|_{[t-r,t]} \\
&\leq \frac{1 + T}{1 - TL}\left(\left\|x^1 - x^2\right\|_{[t-r,t]} + TL\left\|y^1 - y^2\right\|_{[t-r,t]}\right) + \left\|y^1 - y^2\right\|_{[t-r,t]} \\
&= \frac{1 + T}{1 - TL}\left\|x^1 - x^2\right\|_{[t-r,t]} + \frac{1 + T^2L}{1 - TL}\left\|y^1 - y^2\right\|_{[t-r,t]} \\
&\leq \frac{1 + T}{1 - TL}\left(\left\|x^1 - x^2\right\|_{[t-r,t]} + \left\|y^1 - y^2\right\|_{[t-r,t]}\right).
\end{aligned}$$

Hence, we get an estimate for the right-hand upper Dini derivative:

$$
\begin{aligned}
&D^{+}\left(\left\|x_t^1-x_t^2\right\|_{[-r,0]}+\left\|y_t^1-y_t^2\right\|_{[-r,0]}\right) \\
&=\limsup_{T\to 0+}\frac{1}{T}\Big(\left\|x_{t+T}^1-x_{t+T}^2\right\|_{[-r,0]}+\left\|y_{t+T}^1-y_{t+T}^2\right\|_{[-r,0]} \\
&-\left\|x_t^1-x_t^2\right\|_{[-r,0]}-\left\|y_t^1-y_t^2\right\|_{[-r,0]}\Big) \\
&=\limsup_{T\to 0+}\frac{1}{T}\Bigg(\frac{1+T}{1-TL}\left(\left\|x_t^1-x_t^2\right\|_{[-r,0]}+\left\|y_t^1-y_t^2\right\|_{[-r,0]}\right) \\
&-\left\|x_t^1-x_t^2\right\|_{[-r,0]}-\left\|y_t^1-y_t^2\right\|_{[-r,0]}\Bigg) \\
&\leq \limsup_{T\to 0+}\frac{1+L}{1-TL}\left(\left\|x_t^1-x_t^2\right\|_{[-r,0]}+\left\|y_t^1-y_t^2\right\|_{[-r,0]}\right) \\
&\leq (1+L)\left(\left\|x_t^1-x_t^2\right\|_{[-r,0]}+\left\|y_t^1-y_t^2\right\|_{[-r,0]}\right).
\end{aligned}
$$

Then the inequality

$$
D^{+}\left[e^{-(L+1)t}\left(\left\|x_t^1-x_t^2\right\|_{[-r,0]}+\left\|y_t^1-y_t^2\right\|_{[-r,0]}\right)\right]\leq 0
$$

easily follows for all $t\geq 0$. By Zygmund's inequality (see e.g., [34, p. 10] or [24, p. 9]) the function

$$
[0,\infty)\ni t\mapsto e^{-(L+1)t}\left(\left\|x_t^1-x_t^2\right\|_{[-r,0]}+\left\|y_t^1-y_t^2\right\|_{[-r,0]}\right)\in\mathbb{R}
$$

is monotone nonincreasing. Consequently,

$$
\left\|x_t^1-x_t^2\right\|_{[-r,0]}+\left\|y_t^1-y_t^2\right\|_{[-r,0]}\leq e^{(L+1)t}\left(\left\|x_0^1-x_0^2\right\|_{[-r,0]}+\left\|y_0^1-y_0^2\right\|_{[-r,0]}\right)
$$

for all $t\geq 0$. This completes the proof. □

Now, we turn to the study of system (3.1.6), (3.1.2), (3.1.3), and show that it can be considered in the phase space $X\times Z$ as well.

First we show that equation (3.1.3) can be solved uniquely provided $y_t\in Y$.

Proposition 3.3.6. *There is a unique map $\sigma : Y\to Z$ satisfying*

$$
\sigma(\psi)=\frac{1}{c}\psi(-\sigma(\psi)-r_1). \tag{3.3.4}
$$

The map $\sigma : Y\to Z$ is Lipschitz continuous, namely, for all $\psi^1,\psi^2\in Y$ the inequality

$$
\begin{aligned}
\left|\sigma\left(\psi^1\right)-\sigma\left(\psi^2\right)\right| &\leq \frac{1}{a}\max_{s\in[-\max\{\sigma(\psi^1),\sigma(\psi^2)\}-r_1,-r_1]}\left|\psi^1(s)-\psi^2(s)\right| \\
&\leq \frac{1}{a}\left\|\psi^1-\psi^2\right\|_{[-r,0]}
\end{aligned} \tag{3.3.5}
$$

holds.

Proof. Let $\psi \in Y$ be given. Define $\varrho : [0, q/c] \ni s \mapsto s - \psi(-s - r_1)/c$. For $0 \leq s_1 < s_2 \leq q/c$, by using slope$(\psi) \subseteq [a - c, b - c]$ and $0 < a < c$, it follows that

$$\begin{aligned}\frac{\varrho(s_1) - \varrho(s_2)}{s_1 - s_2} &= \frac{s_1 - \psi(-s_1 - r_1)/c - s_2 + \psi(-s_2 - r_1)/c}{s_1 - s_2} \\ &= 1 - \frac{1}{c}\frac{\psi(-s_1 - r_1) - \psi(-s_2 - r_1)}{s_1 - s_2} \\ &= 1 + \frac{1}{c}\frac{\psi(-s_1 - r_1) - \psi(-s_2 - r_1)}{(-s_1 - r_1) - (-s_2 - r_1)} \geq 1 + \frac{a - c}{c} = \frac{a}{c} > 0.\end{aligned}$$

Hence, function ϱ is strictly increasing in $[0, q/c]$. Observe that

$$\varrho(0) = -\frac{\psi(-r_1)}{c} \leq 0 \quad \text{and} \quad \varrho\left(\frac{q}{c}\right) = \frac{q}{c} - \frac{\psi(q/c - r_1)}{c} \geq \frac{q}{c} - \frac{q}{c} = 0.$$

So, ϱ has a unique zero, denoted by $\sigma(\psi)$, in $[0, q/c]$. Clearly, $\sigma(\psi)$ is unique with (3.3.4).

In order to prove the Lipschitz continuity of σ, let ψ^1, ψ^2 be given. If $\sigma(\psi^1) = \sigma(\psi^2)$ then inequality (3.3.5) trivially holds. Without loss of generality assume $\sigma(\psi^1) > \sigma(\psi^2)$. By slope$(\psi^2) \subseteq [a - c, b - c]$ we obtain

$$\begin{aligned}&\left|\sigma\left(\psi^1\right) - \sigma\left(\psi^2\right)\right| = \sigma\left(\psi^1\right) - \sigma\left(\psi^2\right) = \frac{\psi^1(-\sigma(\psi^1) - r_1) - \psi^2(-\sigma(\psi^2) - r_1)}{c} \\ &= \frac{\psi^1(-\sigma(\psi^1) - r_1) - \psi^2(-\sigma(\psi^1) - r_1)}{c} + \frac{\psi^2(-\sigma(\psi^1) - r_1) - \psi^2(-\sigma(\psi^2) - r_1)}{c} \\ &\leq \frac{1}{c}\max_{s \in [-\sigma(\psi^1) - r_1, -r_1]}\left|\psi^1(s) - \psi^2(s)\right| + \frac{c - a}{c}\left|\sigma\left(\psi^1\right) - \sigma\left(\psi^2\right)\right|.\end{aligned}$$

Hence

$$\left(1 - \frac{c - a}{c}\right)\left|\sigma\left(\psi^1\right) - \sigma\left(\psi^2\right)\right| \leq \frac{1}{c}\max_{s \in [-\max\{\sigma(\psi^1), \sigma(\psi^2)\} - r_1, -r_1]}\left|\psi^1(s) - \psi^2(s)\right|,$$

from which inequality (3.3.5) easily holds. □

The next proposition is a key technical result. It shows that, for given $\varphi \in X$ and $\zeta \in Z$, we can find uniquely an element $\psi \in Y$ such that ψ satisfies equation (3.1.2), with $x = \varphi$ and $y = \psi$ a.e. in $[-\zeta - r_1, 0]$, and $\zeta = \sigma(\psi)$ holds as well. In order to guarantee the uniqueness of γ we choose it to have the constant value $c\zeta$ on $[-r, -\zeta - r_1]$.

Proposition 3.3.7. *There is a unique map*

$$\gamma : X \times Z \to Y$$

so that $\psi = \gamma(\varphi, \zeta)$ *satisfies*

$$\begin{cases}\psi(s) = c\zeta \quad \text{for } s \in [-r, -\zeta - r_1], \\ \dot{\psi}(s) = \begin{cases}\varphi(s - r_0) - c & \text{if } 0 < \psi(s) < q, \\ [\varphi(s - r_0) - c]^+ & \text{if } \psi(s) = 0, \\ -[\varphi(s - r_0) - c]^- & \text{if } \psi(s) = q\end{cases} \quad \text{a.e. in } [-\zeta - r_1, 0].\end{cases}$$

In addition, $\zeta = \sigma(\gamma(\varphi, \zeta))$ *for all* $(\varphi, \zeta) \in X \times Z$*, and*

$$\left\|\gamma\left(\varphi^1, \zeta^1\right) - \gamma\left(\varphi^2, \zeta^2\right)\right\|_{[-r,0]} \le r\left\|\varphi^1 - \varphi^2\right\|_{[-r,0]} + \max\{2c-a, b\}\left|\zeta^1 - \zeta^2\right|$$

for all $\varphi^1, \varphi^2 \in X$ *and* $\zeta^1, \zeta^2 \in Z$.

Proof. Let $(\varphi, \zeta) \in X \times Z$ be given. Define a function $\psi : [-r, 0] \to \mathbb{R}$ as follows. Let $\psi(s) = c\zeta$ for $s \in [-r, \zeta - r_1]$. Applying Proposition 3.3.2 with $[t_0, t_1] = [-\zeta - r_1, 0]$, $\xi(s) = \varphi(s - r_0)$ for $s \in [-\zeta - r_1, 0]$, $y^0 = c\zeta$, we can uniquely define $\psi(s) = y(\xi, y^0)(s)$ for $s \in [-\zeta - r_1, 0]$. It is clear that $\gamma(\varphi, \zeta) = \psi$ is the unique element of Y satisfying the stated properties.

By the definition of $\gamma(\varphi, \zeta)$, we have $\zeta = (1/c)\gamma(\varphi, \zeta)(-\zeta - r_1)$, that is $\zeta = \sigma(\gamma(\varphi, \zeta))$.

In order to show the Lipschitz continuity of γ, let $(\varphi^i, \zeta^i) \in X \times Z$ and $\psi^i = \gamma(\varphi^i, \zeta^i)$, $i = 1, 2$. Without loss of generality, assume that $\zeta^1 \ge \zeta^2$. If $-r \le s \le -\zeta^1 - r_1$ then

$$\left|\psi^1(s) - \psi^2(s)\right| = \left|c\zeta^1 - c\zeta^2\right| = c\left|\zeta^1 - \zeta^2\right|. \tag{3.3.6}$$

If $-\zeta^1 - r_1 \le s \le -\zeta^2 - r_1$ then, by using that ψ^1 is absolutely continuous (because it is Lipschitz continuous) and thus $\psi^1(s) - c\zeta^1 = \int_{-\zeta^1 - r_1}^{s} \dot{\psi}^1(u)\, du$,

$$\begin{aligned}
|\psi^1(s) - \psi^2(s)| &= \left|c\zeta^1 + \int_{-\zeta^1 - r_1}^{s} \dot{\psi}^1(u)\, du - c\zeta^2\right| \\
&\le \left|c\zeta^1 - c\zeta^2\right| + \int_{-\zeta^1 - r_1}^{-\zeta^2 - r_1} \left|\varphi^1(u - r_0) - c\right|\, du \\
&\le c\left|\zeta^1 - \zeta^2\right| + \max\{c - a, b - c\}\left|\zeta^1 - \zeta^2\right| = \max\{2c - a, b\}\left|\zeta^1 - \zeta^2\right|.
\end{aligned} \tag{3.3.7}$$

If $-\zeta^2 - r_1 \le s \le 0$ then, similarly to the above case, and applying Claim 3.3.3,

$$\begin{aligned}
\left|\psi^1(s) - \psi^2(s)\right| &= \left|c\zeta^1 + \int_{-\zeta^1 - r_1}^{s} \dot{\psi}^1(u)\, du - c\zeta^2 - \int_{-\zeta^2 - r_1}^{s} \dot{\psi}^2(u)\, du\right| \\
&\le \left|c\zeta^1 - c\zeta^2\right| + \int_{-\zeta^1 - r_1}^{-\zeta^2 - r_1} \left|\varphi^1(u - r_0) - c\right|\, du \\
&\quad + \int_{-\zeta^2 - r_1}^{0} \left|\varphi^1(u - r_0) - \varphi^2(u - r_0)\right|\, du \\
&\le \max\{2c - a, b\}\left|\zeta^1 - \zeta^2\right| + r\left\|\varphi^1 - \varphi^2\right\|_{[-r,0]}.
\end{aligned} \tag{3.3.8}$$

Combining (3.3.6), (3.3.7), (3.3.8), we get the stated Lipschitz continuity. □

Proposition 3.3.8. *Let* $y \in C([-r, \infty), [0, q])$ *be a Lipschitz continuous function with* $\operatorname{slope}(y) \subseteq [a - c, b - c]$. *Then the function* $z : [0, \infty) \ni t \mapsto \sigma(y_t) \in \mathbb{R}$ *satisfies* $\operatorname{slope}(z) \subseteq [1 - c/a, 1 - c/b]$.

Proof. Clearly, $z(t) = \sigma(y_t) = (1/c)y(t-\sigma(y_t)-r_1) = (1/c)y(t-z(t)-r_1)$, $t \geq 0$. Choose $t_1 \geq 0$, $t_2 \geq 0$ with $t_1 \neq t_2$. Then, clearly $t_1 - z(t_1) \neq t_2 - z(t_2)$, and

$$\begin{aligned}\frac{z(t_1)-z(t_2)}{t_1-t_2} &= \frac{1}{c}\frac{y(t_1-z(t_1)-r_1)-y(t_2-z(t_2)-r_1)}{t_1-t_2}\\ &= \frac{1}{c}\frac{y(t_1-z(t_1)-r_1)-y(t_2-z(t_2)-r_1)}{(t_1-z(t_1)-r_1)-(t_2-z(t_2)-r_1)}\,\frac{(t_1-z(t_1)-r_1)-(t_2-z(t_2)-r_1)}{t_1-t_2}\\ &= \frac{1}{c}\frac{y(s_1)-y(s_2)}{s_1-s_2}\left(1-\frac{z(t_1)-z(t_2)}{t_1-t_2}\right)\end{aligned}$$

with $s_j = t_j - z(t_j) - r_1$, $j = 1, 2$. Rearranging terms and multiplying by c we obtain

$$\left(c + \frac{y(s_1)-y(s_2)}{s_1-s_2}\right)\frac{z(t_1)-z(t_2)}{t_1-t_2} = \frac{y(s_1)-y(s_2)}{s_1-s_2}.$$

Using $\operatorname{slope}(y) \subseteq [a-c, b-c]$, and $\xi/(c+\xi) \in [(a-c)/a, (b-c)/b]$ for $\xi \in [a-c, b-c]$, it follows that

$$\frac{z(t_1)-z(t_2)}{t_1-t_2} = \frac{\frac{y(s_1)-y(s_2)}{s_1-s_2}}{c+\frac{y(s_1)-y(s_2)}{s_1-s_2}} \in \left[\frac{a-c}{a}, \frac{b-c}{b}\right] = \left[1-\frac{c}{a}, 1-\frac{c}{b}\right],$$

and the proof is complete. □

Proposition 3.3.9. *Let $y \in C([-r,\infty),[0,q])$ be a Lipschitz continuous function with $\operatorname{slope}(y) \subseteq [a-c, b-c]$, and define $z : [0,\infty) \ni t \mapsto \sigma(y_t) \in \mathbb{R}$. Then the map*

$$\eta : [0,\infty) \ni t \mapsto t - z(t) - r_1 \in \mathbb{R}$$

is Lipschitz continuous with $\operatorname{slope}(\eta) \subseteq [c/b, c/a]$. In particular, η is a strictly increasing function, and, for its inverse η^{-1}, $\operatorname{slope}(\eta^{-1}) \subseteq [a/c, b/c]$ holds.

Proof. From Proposition 3.3.8, with $t_1 \geq 0$, $t_2 \geq 0$ and $t_1 \neq t_2$ we have

$$\frac{\eta(t_1)-\eta(t_2)}{t_1-t_2} = \frac{(t_1-z(t_1)-r_1)-(t_2-z(t_2)-r_1)}{t_1-t_2} = 1 - \frac{z(t_1)-z(t_2)}{t_1-t_2} \in \left[\frac{c}{b}, \frac{c}{a}\right].$$

Let $t_j = \eta(s_j)$, $j = 1, 2$, with $t_1 \neq t_2$. Then, for the inverse

$$\frac{\eta^{-1}(t_1)-\eta^{-1}(t_2)}{t_1-t_2} = \frac{\eta^{-1}(\eta(s_1))-\eta^{-1}(\eta(s_1))}{\eta(s_1)-\eta(s_1)} = \frac{s_1-s_2}{\eta(s_1)-\eta(s_1)} \in \left[\frac{a}{c}, \frac{b}{c}\right],$$

that completes the proof. □

In the remaining part of this section we consider a map $G : X \times Z \to \mathbb{R}$ such that for $F : X \times Y \to \mathbb{R}$ given by

$$F(\varphi, \psi) = G(\varphi, \sigma(\psi))$$

Hypotheses (H1), (H2), (H4) hold. We remark that (H3) also holds with $r_2 = r_1 > 0$ because $\sigma(\psi)$ is determined by $\psi|_{[-r,-r_1]}$. We consider the system composed of equations

$$\dot{x}(t) = G(x_t, \sigma(y_t)) \tag{3.3.9}$$

and (3.1.2) in the phase space $X \times Y$ as it is a particular case of system (3.1.5), (3.1.2).

Define the mappings

$$\begin{aligned} h &: X \times Z \ni (\varphi, \zeta) \mapsto (\varphi, \gamma(\varphi, \zeta)) \in X \times Y, \\ k &: X \times Y \ni (\varphi, \psi) \mapsto (\varphi, \sigma(\psi)) \in X \times Z. \end{aligned} \tag{3.3.10}$$

Note that both of them are Lipschitz continuous, h is injective, but k is not. For their compositions, we have

$$k \circ h = \mathrm{id}_{X\times Z} \quad \text{and} \quad h \circ k\big|_{h(X\times Z)} = \mathrm{id}_{h(X\times Z)}\,.$$

Proposition 3.3.10. *If $\varphi \in X$, $\psi^1 \in Y$, $\psi^2 \in Y$, $\zeta \in Z$ with $\zeta = \sigma(\psi^1) = \sigma(\psi^2)$ and*

$$\psi^1(s) = \psi^2(s) \quad \textit{for all } s \in [-\zeta - r_1, 0],$$

then, for the semiflow Φ generated by system (3.3.9), (3.1.2), we have

$$k\left(\Phi\left(t, \varphi, \psi^1\right)\right) = k\left(\Phi\left(t, \varphi, \psi^2\right)\right) \quad \textit{for all } t \geq 0.$$

Proof. From Theorem 3.3.5 we know that Φ exists. Let $(x^i, y^i) : [-r, \infty) \to \mathbb{R}^2$ be given such that $\Phi\left(t, \varphi^i, \psi^i\right) = (x^i_t, y^i_t)$, $t \geq 0)$.

First we show that

$$x^1(t) = x^2(t) \text{ for all } t \in [-r, \infty), \quad y^1(t) = y^2(t) \text{ for all } t \in [-\zeta - r_1, \infty). \tag{3.3.11}$$

If (3.3.11) does not hold, then there exists a maximal $t_0 \in [0, \infty)$ such that

$$x^1(t) = x^2(t) \text{ for all } t \in [-r, t_0], \quad y^1(t) = y^2(t) \text{ for all } t \in [-\zeta - r_1, t_0]. \tag{3.3.12}$$

We claim that $\sigma\left(y^1_t\right) = \sigma\left(y^2_t\right)$ for all $t \in [0, t_0 + r_1]$.

Proposition 3.3.9 implies, for $i = 1, 2$, that

$$t - \sigma\left(y^i_t\right) - r_1 \geq -\sigma\left(y^i_0\right) - r_1 = -\zeta - r_1 \quad \text{for all } t \in [0, \infty).$$

From this inequality and from Proposition 3.3.6, it follows for each $t \in [0, t_0 + r_1]$ that

$$\begin{aligned} &\left|\sigma\left(y^1_t\right) - \sigma\left(y^2_t\right)\right| \leq \frac{1}{a} \max_{s \in \left[-\max\left\{\sigma\left(y^1_t\right), \sigma\left(y^2_t\right)\right\} - r_1, -r_1\right]} \left|y^1(t+s) - y^2(t+s)\right| \\ &= \frac{1}{a} \max_{s \in \left[t - \max\left\{\sigma\left(y^1_t\right), \sigma\left(y^2_t\right)\right\} - r_1, t - r_1\right]} \left|y^1(s) - y^2(s)\right| \leq \frac{1}{a} \max_{s \in [-\zeta - r_1, t_0]} \left|y^1(s) - y^2(s)\right| = 0. \end{aligned}$$

Therefore the claim holds.

Set $m(t) = \sigma\left(y^1_t\right) = \sigma\left(y^2_t\right)$, $t \in [0, t_0 + r_1]$. Clearly, for both x^1 and x^2 the same equation $\dot{x}(t) = G(x_t, m(t))$ holds for all $t \in (0, t_0 + r_1)$. By (3.3.12), $x^1_t = x^2_t$ for all $t \in [0, t_0]$. Since $G : X \times Z \to \mathbb{R}$ is Lipschitz continuous, a standard uniqueness technique yields the existence of a $\delta \in (0, r_1)$ so that $x^1(t) = x^2(t)$ for all $t \in [-r, t_0 + \delta]$. Now we can apply Proposition 3.3.2 to conclude $y^1(t) = y^2(t)$ for all $t \in [-\zeta - r_1, t_0 + \delta]$. This contradicts the definition of t_0. It follows that (3.3.11) is satisfied, and also $k(x^1_t, y^1_t) = k(x^2_t, y^2_t)$ for all $t \geq 0$. This proves our statement. □

Now we have all tools to show that for system (3.1.6), (3.1.2), (3.1.3) the space $X \times Z$ is a suitable phase space.

Theorem 3.3.11. *For each* $(\varphi, \zeta) \in X \times Z$ *there exists a unique pair of functions*

$$x^{\varphi,\zeta} : [-r, \infty) \to \mathbb{R}, \qquad z^{\varphi,\zeta} : [0, \infty) \to \mathbb{R}$$

such that (x, z) *is a solution of system* (3.1.6), (3.1.2), (3.1.3) *in the phase space* $X \times Z$ *satisfying the initial condition* $x_0^{\varphi,\zeta} = \varphi$, $z^{\varphi,\zeta}(0) = \zeta$*. The mapping*

$$\Psi : [0, \infty) \times X \times Z \ni (t, \varphi, \zeta) \mapsto \left(x_t^{\varphi,\zeta}, z^{\varphi,\zeta}(t)\right) \in X \times Z$$

defines a continuous semiflow on $X \times Z$*. In addition, there exists a constant* $M > 0$ *such that*

$$\left\|\Psi\left(t, \varphi^1, \zeta^1\right) - \Psi\left(t, \varphi^2, \zeta^2\right)\right\| \leq M \left\|\left(\varphi^1, \zeta^1\right) - \left(\varphi^2, \zeta^2\right)\right\| e^{t(1+L)}$$

for all $t \geq 0$, $\varphi^1, \varphi^2 \in X$, $\zeta^1, \zeta^2 \in Z$*. Moreover, for all* $t \geq 0$, $\varphi \in X$, $\zeta \in Z$,

$$\Psi(t, \varphi, \zeta) = k(\Phi(t, h(\varphi, \zeta))).$$

Proof. Let $(\varphi, \zeta) \in X \times Z$ be given.

1. Existence. By Theorem 3.3.5, system (3.3.9), (3.1.2) has a unique solution in the phase space $X \times Y$, denoted by $(x, y) : [-r, \infty) \to \mathbb{R}^2$, with $x_0 = \varphi$, $y_0 = \gamma(\varphi, \zeta)$. Defining $z(t) = \sigma(y_t) \in Z$, $t \in [0, \infty)$, (x, z) is a solution of system (3.1.6), (3.1.2), (3.1.3) in $X \times Z$ with $x_0 = \varphi$, $z(0) = \zeta$.

2. Uniqueness. Assume that the pair of functions $\widetilde{x} : [-r, \omega) \to \mathbb{R}$, $\widetilde{z} : [0, \omega) \to \mathbb{R}$ is a also solution of system (3.1.6), (3.1.2), (3.1.3) in $X \times Z$ with initial condition $\widetilde{x}_0 = \varphi$, $\widetilde{z}(0) = \zeta$. Then, by definition, there exists a Lipschitz continuous function $\widetilde{y} : [-r, \omega) \to \mathbb{R}$ so that $\widetilde{y}_t \in Y$, $\widetilde{z}(t) = \sigma(\widetilde{y}_t)$ for all $t \in [0, \omega)$, and equation (3.1.2) holds a.e. in $[-\zeta - r_1, \omega)$. From $\widetilde{z}(0) = \zeta = \sigma(\widetilde{y}_0) = (1/c)\widetilde{y}(-\zeta - r_1)$ and (3.1.2), it easily follows that

$$\widetilde{y}(s) = \gamma(\varphi, \zeta)(s) \quad \text{for all } s \in [-\zeta - r_1, 0].$$

Proposition 3.3.10 implies that

$$(x_t, z(t)) = k(\Phi(t, \varphi, \gamma(\varphi, \zeta))) = k(\Phi(t, \varphi, \widetilde{y}_0)) = (\widetilde{x}_t, \widetilde{z}(t)) \quad \text{for all } t \in [0, \omega).$$

It is clear that the pair $(\widetilde{x}, \widetilde{y}) : [-r, \omega) \to \mathbb{R}^2$ is also a solution of system (3.3.9), (3.1.2) in $X \times Y$. Moreover $\Phi(t, \varphi, \widetilde{y}_0)$ can be uniquely extended to $[0, \infty)$ with $\Phi(t, \varphi, \widetilde{y}_0) = (\widetilde{x}_t, \widetilde{y}_t)$ for all $t \in [0, \omega)$.

Now we see that the function Ψ is well defined.

3. Properties of Ψ. We have to show that Ψ is a semiflow on $X \times Z$, i.e.,

$$\Psi(t_1 + t_2, \varphi, \zeta) = \Psi(t_2, \Psi(t_1, \varphi, \zeta)) \quad \text{for all } t_1 \geq 0,\ t_2 \geq 0. \tag{3.3.13}$$

Let $x : [-r, \infty) \to \mathbb{R}$, $z : [-\zeta - 1, \infty) \to \mathbb{R}$ be the solution of system (3.1.6), (3.1.2), (3.1.3), $y : [-r, \infty) \to \mathbb{R}$ be such that (3.1.2) holds a.e. in $[-r, \infty)$, and $t_1 \geq 0$, $t_2 \geq 0$.

Since $\Psi(t, \varphi, \zeta) = (x_t, z(t))$ for $t \geq 0$, (3.3.13) is equivalent to

$$(x_{t_1+t_2}, z(t_1 + t_2)) = \Psi(t_2, (x_{t_1}, z(t_1))). \tag{3.3.14}$$

Using functions h and k defined in (3.3.10), it is easy to see that

$$\Psi(t, \varphi, \zeta) = k(\Phi(t, h(\varphi, \zeta))) \quad \text{for all } t \geq 0,$$

so (3.3.14) can be written as

$$(x_{t_1+t_2}, z(t_1 + t_2)) = k(\Phi(t_2, x_{t_1}, \gamma(x_{t_1}, z(t_1)))). \tag{3.3.15}$$

The assumptions of Proposition 3.3.10 clearly hold with x_{t_1}, $\gamma(x_{t_1}, z(t_1))$, y_{t_1}, z_{t_1} instead of φ, ψ^1, ψ^2, ζ, so we have

$$\begin{aligned} k(\Phi(t_2, x_{t_1}, \gamma(x_{t_1}, z(t_1)))) &= k(\Phi(t_2, x_{t_1}, y_{t_1})) = k(x_{t_1+t_2}, y_{t_1+t_2}) \\ &= (x_{t_1+t_2}, \sigma(y_{t_1+t_2})) = (x_{t_1+t_2}, z(t_1 + t_2)). \end{aligned}$$

Thus, (3.3.15) holds, and Ψ is a semiflow on $X \times Z$.

From the Lipschitz property of Φ in Theorem 3.3.5, and the Lipschitz continuity of h and k, our lasts statement also holds, where M is the product of the Lipschitz constants of h and k. □

3.4 Slowly oscillating periodic solutions

In this section it is shown that, for a class of rate control problems, like the system composed of equations (3.1.4), (3.1.3), (3.1.2), it is possible to have slowly oscillatory periodic solutions.

We need the simplifying assumption $r_0 = 0$. This condition is important technically. Equation (3.1.4) with $r_0 = 0$ becomes an equation with a single delay, while in case $r_0 > 0$ there are two different delays. By rescaling the time, without loss of generality we may suppose $r_1 = 1$.

Recall from Section 3.1 the constants a, b, c, q, with $0 < a < c < b$, $q > 0$. For the rate $x(t)$, $x(t) \in [a, b]$ is assumed, c is the maximal capacity of the server, q is an upper bound for the length of the queue $y(t)$. We suppose that there exists $x_* \in (a, c)$ serving as a stationary solution of the rate control equation.

As we are interested in the oscillatory behaviour of $x(t)$ around x_* we introduce $v(t) = x(t) - x_*$, and write the rate control equation for v instead of x. Defining $d = c - x_* > 0$, we obtain system (3.1.7), (3.1.8), (3.1.9), which will be studied in this section.

Set $A = a - x_*$, $B = b - x_*$, and assume the following conditions for f and g:

(S1) $f, g \in C^1([A,B],\mathbb{R})$;

(S2) $f(\xi)\xi \geq 0$ and $g(\xi)\xi > 0$ for all $\xi \in [A,B] \setminus \{0\}$, $g'(0) > 0$;

(S3) $g([A,B]) \in (-f(B), -f(A))$;

(S4) the map $\mathbb{C} \ni \lambda \mapsto \lambda + f'(0) + g'(0)e^{-\lambda} \in \mathbb{C}$ has a zero with positive real part.

Define the functions $\widetilde{f}, \widetilde{g} : [A,B] \to \mathbb{R}$ as follows:

$$\widetilde{f}(\xi) = \begin{cases} \frac{f(\xi)}{\xi} & \text{if } \xi \neq 0, \\ f'(0) & \text{if } \xi = 0, \end{cases} \qquad \widetilde{g}(\xi) = \begin{cases} \frac{g(\xi)}{\xi} & \text{if } \xi \neq 0, \\ g'(0) & \text{if } \xi = 0. \end{cases}$$

From (S1) and (S2) it follows that $\widetilde{f}$ and $\widetilde{g}$ are continuous, and there are constants $f_1 \geq 0$, $g_1 > g_0 > 0$ such that

$$\widetilde{f}([A,B]) \subseteq [0, f_1], \quad \widetilde{g}([A,B]) \subseteq [g_0, g_1].$$

Let

$$K_0 = (f_1 + g_1)\max\{-A, B\}, \quad r = 1 + \frac{q}{c}, \quad K_1 = rK_0.$$

Since a, b, c, q, r are given, by setting $K = K_1$, we can define the sets X and Y as in Section 3.1. It is easy to verify that system (3.1.7), (3.1.8), (3.1.9) is a particular case of system (3.1.6), (3.1.2), (3.1.3) when $r_0 = 0$, $r_1 = 1$,

$$G(\varphi, \zeta) = -f(\varphi(0) - x_*) - g(\varphi(-\zeta - 1) - x_*) \quad ((\varphi, \zeta) \in X \times Z),$$

and $x(t) = v(t) + x_*$. Moreover, under Hypotheses (S1)–(S3), with $F(\varphi, \psi) = G(\varphi, \sigma(\psi))$, (H1)–(H4) hold. Therefore, by Theorem 3.3.11, for all $(\varphi, \zeta) \in X \times Z$, there exists a unique solution $x^{\varphi,\zeta} : [-r, \infty) \to \mathbb{R}$, $z^{\varphi,\zeta} : [0, \infty) \to \mathbb{R}$ with $x_0^{\varphi,\zeta} = \varphi$, $z^{\varphi,\zeta}(0) = \zeta$, and $\Psi(t, \varphi, \zeta) = (x_t^{\varphi,\zeta}, z^{\varphi,\zeta}(t))$.

For $\varphi \in C_{[-r,0]}$ and $k \in \mathbb{R}$ define $\varphi + k \in C_{[-r,0]}$ as $[-r, 0] \ni s \mapsto \varphi(s) + k \in \mathbb{R}$. Introduce the set

$$\mathcal{X} = \left\{\varphi \in C_{[-r,0]} \;\middle|\; \varphi([-r,0]) \subseteq [A,B],\ \mathrm{lip}(\varphi) \leq K_1\right\} = \left\{\varphi \in C_{[-r,0]} \;\middle|\; \varphi + x_* \in X\right\}.$$

Now, for each $(\varphi, \zeta) \in \mathcal{X} \times Z$, the unique solution $v = v^{\varphi,\zeta} : [-r, \infty) \to \mathbb{R}$, $z = z^{\varphi,\zeta} : [0, \infty) \to \mathbb{R}$ of system (3.1.7), (3.1.8), (3.1.9) with $v_0 = \varphi$, $z(0) = \zeta$ can be determined as

$$\left(v_t^{\varphi,\zeta}, z^{\varphi,\zeta}(t)\right) = \Psi(t, \varphi + x_*, \zeta) - (x_*, 0). \tag{3.4.1}$$

In addition to v and z, there exists $y = y^{\varphi,\zeta} : [-r, \infty) \to \mathbb{R}$ with $y_t \in Y$ for all $t \geq 0$, and y is uniquely determined on $[-\zeta - 1, \infty)$ such that equation (3.1.7) holds for all $t > 0$, (3.1.9) holds for all $t \geq 0$, and (3.1.8) holds almost everywhere on $[-\zeta - 1, \infty)$. Therefore, we obtain

Proposition 3.4.1. *Under Conditions (S1)–(S3), the solutions of system* (3.1.7), (3.1.8), (3.1.9) *define a continuous semiflow by* (3.4.1) *on* $\mathcal{X} \times Z$*, and the same Lipschitz continuity holds for the semiflow as for* Ψ *in Theorem 3.3.11.*

In the sequel, when we write (v, z, y), we always mean that $v = v^{\varphi,\zeta}$, $z = z^{\varphi,\zeta}$, $y = y^{\varphi,\zeta}$ for some $(\varphi, \zeta) \in \mathcal{X} \times Z$.

Define $T_0 = 2q/d$.

Proposition 3.4.2. *If* $\tau_1 \geq -\zeta - 1$, $\tau_2 \geq \tau_1 + T_0$, *and* $v(t) \leq d/2$ *for all* $t \in [\tau_1, \tau_2]$, *then* $y(t) = 0$ *for all* $t \in [\tau_1 + T_0, \tau_2]$. *If, in addition,* $\tau_2 \geq \tau_1 + T_0 + 1$, *then* $z(t) = 0$ *for all* $t \in [\tau_1 + T_0 + 1, \tau_2]$.

Proof. From equation (3.1.8) and from $v(t) \leq d/2$, $t \in [\tau_1, \tau_2]$, it follows that, if there is $\tau_* \in [\tau_1, \tau_2)$ with $y(\tau_*) = 0$, then $\dot{y}(t) \leq 0$ almost everywhere in $[\tau_*, \tau_2]$, and thus, $y(t) = 0$ for all $t \in [\tau_*, \tau_2]$. Consequently, either $y(t) = 0$ for all $t \in [\tau_1, \tau_2]$, or there exists a maximal $\tau_{**} \in (\tau_1, \tau_2]$ with $y(t) > 0$ for all $t \in [\tau_1, \tau_{**})$. In the first case the statements of the proposition trivially hold. In the second case, by equation (3.1.8), $\dot{y}(t) = v(t) - d \leq -d/2$ almost everywhere in $[\tau_1, \tau_{**}]$. As $y(\tau_1) \in [0, q]$, it easily follows that $0 \leq y(\tau_{**}) \leq q - (d/2)(\tau_{**} - \tau_1)$, and hence $\tau_{**} \leq \tau_1 + T_0$. Therefore, $y(t) = 0$ for all $t \in [\tau_1 + T_0, \tau_2]$. The statement for z can be obtained by using equation (3.1.9). □

Observe that $(0, 0) \in \mathcal{X} \times Z$ is a stationary point of the semiflow generated by system (3.1.7), (3.1.8), (3.1.9). Under Conditions (S1)–(S3), and assuming that (S4) does not hold, and slightly more, that is

(S5) $\operatorname{Re} z < 0$ for all zeros of the map $\mathbb{C} \ni \lambda \mapsto \lambda + f'(0) + g'(0)e^{-\lambda} \in \mathbb{C}$,

it is expected that $(0, 0)$ is stable. In fact, combining Theorem 3.3.11 and Proposition 3.4.2, local stability is straightforward.

Theorem 3.4.3. *Assume that Conditions (S1)–(S3), (S5) hold. Then the stationary point* $(0, 0) \in \mathcal{X} \times Z$ *of the semiflow generated by system* (3.1.7), (3.1.8), (3.1.9) *is locally asymptotically stable.*

Proof. By Theorem 3.3.11 there exists $\widetilde{L} > 1$ such that, for each $(\varphi, \zeta) \in \mathcal{X} \times Z$ the unique solution $v = v^{\varphi,\zeta}$, $z = z^{\varphi,\zeta}$ of system (3.1.7), (3.1.8), (3.1.9) satisfies

$$\begin{aligned} \|(v_t, z(t)) - (0, 0)\| &= \|\Psi(t, \varphi + x_*, \zeta) - (x_*, 0)\| \\ &= \|\Psi(t, \varphi + x_*, \zeta) - \Psi(t, x_*, 0)\| \leq \widetilde{L}\|(\varphi, \zeta)\| \end{aligned}$$

for all $t \in [0, T_0 + 1]$.

As Proposition 3.4.2 holds with $\tau_1 = -1$ and arbitrarily large τ_2, if $v(t) \leq d/2$ for all $t \geq -r$, then $z(t) = 0$ for all $t \geq -1 + T_0 + 1 = T_0$, and, consequently,

$$\dot{v}(t) = -f(v(t)) - g(v(t - 1)) \quad \text{for all } t > T_0 + 1.$$

A classical result for equations with constant delay (see e.g. [12, 16]) is that, under Conditions (S1)–(S3), (S5), for each $\varepsilon \in (0, d/2)$ there exists $\delta = \delta(\varepsilon) \in (0, \varepsilon)$ such that

$$\max_{-1 \le s \le 0} |v(T_0 + 1 + s)| \le \delta \;\text{ implies }\; |v(t)| < \varepsilon \;\text{ for all }\; t \ge T_0.$$

Now, for given $\varepsilon \in (0, d/2)$ choosing $(\varphi, \zeta) \in \mathcal{X} \times Z$ with $\|(\varphi, \zeta)\| < \delta(\varepsilon)/\widetilde{L}$, it should be clear that $\|(v_t, z(t))\| < \varepsilon$ follows for all $t \ge 0$. That is, $(0, 0)$ is locally stable.

Asymptotic stability follows in the same way by using again the constant delay result from [12]. □

From this point throughout this section, we assume that Conditions (S1)–(S4) are satisfied. Then instability of $(0, 0) \in \mathcal{X} \times Z$ can be easily obtained. We show after a series of technical results that there exists a nontrivial slowly oscillating periodic solution (v, z) of system (3.1.7), (3.1.8), (3.1.9). Here slow oscillation of (v, z) means that

$$t_1 < t_2 - z(t_2) - 1$$

holds for any two zeros $t_1 < t_2$ of v.

Observe that equation (3.1.7) can be written as

$$\dot{v}(t) = -\widetilde{f}(v(t))v(t) - \widetilde{g}(v(t - z(t) - 1))v(t - z(t) - 1). \tag{3.4.2}$$

For $(\varphi, \zeta) \in \mathcal{X} \times Z$ consider $v = v^{\varphi,\zeta}$, $z = z^{\varphi,\zeta}$. Define

$$u = u^{\varphi,\zeta} : [-r, \infty) \ni t \mapsto v(t) \exp\left(\int_0^t \widetilde{f}(v(s))\, ds\right) \in \mathbb{R} \qquad \text{and}$$
$$C : [0, \infty) \ni t \mapsto \widetilde{g}(v(t - z(t) - 1)) \exp\left(\int_{t-z(t)-1}^t \widetilde{f}(v(s))\, ds\right) \in \mathbb{R}$$

where $z = z^{\varphi,\zeta}$. Setting $c_0 = g_0$, $c_1 = g_1 e^{f_1 r}$, for all $(\varphi, \zeta) \in \mathcal{X} \times Z$ we have

$$C(t) \in [c_0, c_1] \quad \text{for all } t \ge 0.$$

Proposition 3.4.4. *For each $(\varphi, \zeta) \in \mathcal{X} \times Z$, the function $u = u^{\varphi,\zeta} : [-r, \infty) \to \mathbb{R}$ is continuous, it is continuously differentiable on $(0, \infty)$, and*

$$\dot{u}(t) = -C(t)u(t - z(t) - 1) \qquad (t > 0) \tag{3.4.3}$$

holds with $z = z^{\varphi,\zeta}$. In addition,

$$|u(t)| \le |v(t)| \le |u(t)|e^{f_1 r} \quad \text{for all } t \in [-r, 0],$$
$$|v(t)| \le |u(t)| \le |v(t)|e^{f_1 t} \quad \text{for all } t \ge 0.$$

Proof. Differentiating u and using equation (3.4.2) for $t > 0$, we get

$$\begin{aligned}\dot{u}(t) &= \Big(\dot{v}(t) + \widetilde{f}(v(t))\Big) \exp\left(\int_0^t \widetilde{f}(v(s))\, ds\right)\\ &= -\widetilde{g}(v(t - z(t) - 1))v(t - z(t) - 1)\\ &\cdot \exp\left(\int_0^{t-z(t)-1} \widetilde{f}(v(s))\, ds\right) \exp\left(\int_{t-z(t)-1}^{t} \widetilde{f}(v(s))\, ds\right)\\ &= -C(t)u(t - z(t) - 1),\end{aligned}$$

so equation (3.4.3) holds. The continuity and differentiability property of u is immediate from the definition. The stated inequalities between $|u(t)|$ and $|v(t)|$ are easy consequences of the definition and the bounds on $\widetilde{f}$. □

Let

$$W = \Big\{(\varphi, \zeta) \in \mathcal{X} \times Z \;\Big|\; \varphi(s) = 0 \text{ for all } s \in [-r, -1 - \zeta],$$
$$[-\zeta - 1] \ni s \mapsto \varphi(s)e^{f_1 s} \in \mathbb{R} \text{ is nondecreasing, } \varphi(0) > 0\Big\}$$

and $W_0 = W \cup \{(0, 0)\}$. Our plan is to define a return map on W_0 and to show that it has a nontrivial fixed point on W_0 corresponding to a slowly oscillating periodic orbit.

Proposition 3.4.5. *There exists a constant $T_2 > 1$ such that for all $(\varphi, \zeta) \in W$, $v = v^{\varphi,\zeta}$ has at least two zeros in $[0, T_2]$.*

Proof. As $v = v^{\varphi,\zeta}$ and $u = u^{\varphi,\zeta}$ have the same zeros, it suffices to show the statement for $u = u^{\varphi,\zeta}$.

Let λ be a zero of $\lambda + f'(0) + g'(0)e^{-\lambda}$ with $\operatorname{Re}\lambda > 0$ guaranteed by Hypothesis (S4). Setting $\mu = \lambda + f'(0)$, we have $\operatorname{Re}\mu > 0$ and $\mu + g'(0)e^{f'(0)}e^{-\mu} = 0$. This is possible only if

$$g'(0)e^{f'(0)} > \frac{\pi}{2}$$

(see [12, Ch. XI.]). As $\widetilde{f}$, $\widetilde{g}$ are continuous and $\widetilde{f}(0) = f'(0)$, $\widetilde{g}(0) = g'(0)$, there exists $\delta \in (0, d/2)$ such that

$$\widetilde{g}(\xi_1)e^{\widetilde{f}(\xi_2)} > \frac{\pi}{2} \quad \text{for } |\xi_1| \le \delta,\ |\xi_2| \le \delta.$$

Observe that $B/\delta > 1$. Define

$$s_0 = r + \frac{1}{c_0} \log \frac{Be^{f_1 r}}{\delta}, \quad s_1 = s_0 + T_0 + 1, \quad T_1 = s_1 + 7.$$

First, we prove that for all $(\varphi, \zeta) \in W$, $u = u^{\varphi,\zeta}$ has at least one zero in $[0, T_1]$. Indirectly, assume that there exists a $(\varphi, \zeta) \in W$ such that $u(t) > 0$ for all $t \in [0, T_1]$. By the definition of W and our assumption, u is nonnegative on $[-r, T_1]$. From Proposition 3.4.4 and equation (3.4.3) it follows that $\dot{u}(t) \leq 0$ for all $t \in (0, T_1]$. Thus, u is monotone nonincreasing on $[0, T_1]$. In particular, $u(t) \leq u(t - z(t) - 1)$ for $t \in [r, T_1]$. Then, again by Proposition 3.4.4,

$$\dot{u}(t) \leq -c_0 u(t) \quad \text{for all } t \in [r, T_1]. \tag{3.4.4}$$

As $v(r) \leq B$, $u(r) \leq Be^{f_1 r}$, from inequality (3.4.4) we get

$$u(t) \leq Be^{f_1 r} e^{-c_0(t-r)} \quad \text{for all } t \in [r, T_1].$$

Then, since $Be^{f_1 r} e^{-c_0(s_0 - r)} = \delta$, for all $t \in [s_0, T_1]$,

$$v(t) \leq u(t) \leq Be^{f_1 r} e^{-c_0(t-r)} \leq Be^{f_1 r} e^{-c_0(s_0-r)} = \delta < \frac{d}{2}.$$

Applying Proposition 3.4.2 with $\tau_1 = s_0$, $\tau_2 = T_1$, we find $z(t) = 0$ for all $t \in [s_1, T_1]$. This means that equation (3.4.3) becomes

$$\dot{u}(t) = -C(t)u(t-1) \quad \text{for all } t \in [s_1, T_1]$$

where, by $v(t) \leq \delta$ for all $t \in [s_0, T_1]$, and by the choice of δ,

$$C(t) = \widetilde{g}(v(t-1)) \exp\left(\int_{t-1}^{t} \widetilde{f}(v(s))\, ds\right) \geq \widetilde{g}(v(t-1)) \exp\left(\min_{s \in [-1,0]} \widetilde{f}(v(s))\right) > \frac{\pi}{2}.$$

There exists a minimal integer $N \geq 1$ with $4N \geq s_1 + 1$. Clearly, $4N \leq s_1 + 5$ and $4N + 2 \leq T_1 = s_1 + 7$. The function $\sin(\pi/2)t$ is positive on $(4N, 4N+2)$, has zero at $4N$ and $4N + 2$. Define

$$w_\varepsilon(t) = \varepsilon \sin\left(\frac{\pi}{2} t\right), \quad \varepsilon > 0,\ t \in \mathbb{R}.$$

As u is positive on $[4N, 4N+2]$, there are a minimal $\varepsilon = \varepsilon_0 > 0$ such that $w(t) = w_{\varepsilon_0}(t) \leq u(t)$ for all $t \in [4N, 4N+2]$, and a minimal $t \in (4N, 4N+2)$, denoted by $\hat{t}$ with $w(\hat{t}) = u(\hat{t})$. Now it is clear that

$$\dot{w}\left(\hat{t}\right) = \dot{u}\left(\hat{t}\right), \quad \text{and} \quad w(t) < u(t) \ \text{ for all } t \in \left[4N, \hat{t}\right).$$

From the monotonicity of u on $[0, T_1]$, it follows that $\dot{w}(\hat{t}) = \dot{u}\left(\hat{t}\right) \leq 0$. Consequently,

$$\hat{t} \in [4N+1, 4N+2) \quad \text{and} \quad \hat{t} - 1 \in [4N, 4N+1).$$

Hence we obtain $0 \leq w(\hat{t} - 1) < u(\hat{t} - 1)$.

Therefore, by using $C(\hat{t}) > \pi/2$, $\dot{w}(t) = -(\pi/2)w(t-1)$ and $0 \leq w(\hat{t} - 1) < u(\hat{t} - 1)$, we get

$$\dot{u}\left(\hat{t}\right) = -C\left(\hat{t}\right) u\left(\hat{t} - 1\right) < -\frac{\pi}{2} w\left(\hat{t} - 1\right) = \dot{w}\left(\hat{t}\right),$$

a contradiction to $\dot{u}(\hat{t}) = \dot{w}(\hat{t})$. Thus, u has a zero t^* in $[0, T_1]$. By similar argument, we can show that there exists a constant $s_2 > 0$ such u has another zero in $(t^*, t^* + s_2 + 7]$. Thus the statement is true with $T_2 = s_1 + s_2 + 14$. □

Let $(\varphi,\zeta)\in W$, $v=v^{\varphi,\zeta}$, $z=z^{\varphi,\zeta}$, $u=u^{\varphi,\zeta}$. Proposition 3.4.5 allows us to define $t_0\in[-r,-1]$, $t_1,t_2\in(0,T_2]$ as

$$\begin{aligned}t_0&=t_0(\zeta)=-\zeta-1,\\ t_1&=t_1(\varphi,\zeta)=\min\{t>0\mid v(t)=0\},\\ t_2&=t_2(\varphi,\zeta)=\min\{t>t_1\mid v(t)=0\}.\end{aligned}$$

Recall the function $\eta=\eta^{\varphi,\zeta}:[0,\infty)\ni t\mapsto t-z(t)-1\in\mathbb{R}$ from Proposition 3.3.9 and its properties $\text{slope}(\eta)\subseteq[c/b,c/a]$, η^{-1} exists and $\text{slope}(\eta^{-1})\subseteq[a/c,b/c]$. Then we can define

$$t_0^*=\eta^{-1}(t_0)=0,\quad t_1^*=t_1^*(\varphi,\zeta)=\eta^{-1}(t_1),\quad t_2^*=t_2^*(\varphi,\zeta)=\eta^{-1}(t_2).$$

Clearly, $t_j^*\in[0,T_2+r]$ for $j\in\{0,1,2\}$. From equation (3.1.7), Conditions (S1)–(S4) and the above definitions it easily follows that the map $[-r,0]\ni s\mapsto v(s)e^{f_1 s}\in\mathbb{R}$ is monotone nondecreasing, $v|_{[-r-\zeta-1]}=0$, v is positive on $[0,t_1)$ and on $(t_2,t_2^*]$, and it is negative on (t_1,t_2), see Figure 3.2. The function u is nonnegative on $[-r,0]$, it is positive on $[0,t_1)$ and on $(t_2,t_2^*]$, and it is negative on (t_1,t_2), moreover it is monotone nonincreasing on $[0,t_1^*]$, and monotone increasing on $[t_1^*,t_2^*]$. In particular, we have

$$-r\le t_0=-\zeta-1<t_0^*=0<t_1<t_1^*<t_2<t_2^*\le T_2+r.$$

Proposition 3.4.6. *The functions*

$$W\ni(\varphi,\zeta)\mapsto t_j(\varphi,\zeta)\in[-r,T_2],\qquad W\ni(\varphi,\zeta)\mapsto t_j^*(\varphi,\zeta)\in[0,T_2+r]$$

are continuous for $j\in\{0,1,2\}$.

Proof. The statement is evident for $j=0$. Let $(\varphi,\zeta)\in W$ and a sequence $(\varphi^n,\zeta^n)_{n=0}^\infty$ in W be given with $(\varphi^n,\zeta^n)\to(\varphi,\zeta)$ as $n\to\infty$ in the norm of $C_{[-r,0]}\times\mathbb{R}$. Theorem 3.3.11 implies, with the notation $v=v^{\varphi,\zeta}$, $z=z^{\varphi,\zeta}$, $v^n=v^{\varphi^n,\zeta^n}$, $z^n=z^{\varphi^n,\zeta^n}$, that

$$\begin{aligned}&v^n(t)\to v(t)\text{ as }n\to\infty\text{ uniformly in }t\in[-r,T_2+r],\\ &z^n(t)\to z(t)\text{ as }n\to\infty\text{ uniformly in }t\in[0,T_2+r].\end{aligned}\tag{3.4.5}$$

Then the right hand side of equation (3.1.7) with $v=v^n,\ z=z^n$ tends to the right hand side of (3.1.7) as $n\to\infty$ uniformly in $t\in[0,t_2+r]$. Consequently,

$$\dot v^n(t)\to\dot v(t)\text{ as }n\to\infty\text{ uniformly in }t\in(0,T_2+r].$$

It is elementary to show that these uniform convergences guarantee the continuity of $t_1(\varphi,\zeta)$ and $t_2(\varphi,\zeta)$ in (φ,ζ) provided that t_1 and t_2 are simple zeros.

It is clear that $\dot v(t_1)\le 0$. If $\dot v(t_1)=0$ then, by equation (3.1.7) and $v(t_1)=0$, $g(v(t_1-z(t_1)-1))=0$ and $v(t_1-z(t_1)-1)=0$ follow. The minimality of the zero t_1 in $[0,T_2]$ yields $t_1-z(t_1)-1<0$. Hence, by the definition of W and $(\varphi,\zeta)\in W$, $v_0=\varphi$, one finds $v(t) = 0$ for all $t \in [-r, t_1 - z(t_1) - 1]$. Using the monotone increasing property of $t \mapsto t - z(t) - 1$, we conclude $\dot{v}(t) = -f(v(t))$ for all $t \in (0, t_1]$. This is an ordinary differential equation, so it is uniquely solvable backwards. As $v(t_1) = 0$, $f(0) = 0$, it gives $v(t) = 0$ for $t \in (0, t_1]$. By continuity, we get a contradiction. Therefore, $\dot{v}(t_1) < 0$, and t_1 is a simple zero of v.

For t_2 we have $t_2 - z(t_2) - 1 \in (t_1, t_2)$, and thus $v(t_2 - z(t_2) - 1) < 0$. Hence $\dot{v}(t_2) = -g(v(t_2 - z(t_2) - 1)) > 0$. Therefore, $t_1(\varphi, \zeta)$ and $t_2(\varphi, \zeta)$ are continuous in $(\varphi, \zeta) \in W$.

It also follows from (3.4.5) that

$$\eta^n(t) \to \eta(t) \text{ as } n \to \infty \text{ uniformly in } t \in [0, T_2 + r], \tag{3.4.6}$$

where $\eta = \eta^{\varphi,\zeta}$, $\eta^n = \eta^{\varphi^n,\zeta^n}$. Define $t_1^n = t_1(\varphi^n, \zeta^n)$ and $t_1^{n,*} = (\eta^n)^{-1}(t_1^n)$. From $t_1 = \eta(t_1^*)$, $t_1^n = \eta^n(t_1^{n,*})$ and the Lipschitz property of η, one obtains

$$\begin{aligned} |t_1 - t_1^n| &= |\eta(t_1^*) - \eta^n(t_1^{n,*})| \geq |\eta(t_1^*) - \eta(t_1^{n,*})| - |\eta(t_1^{n,*}) - \eta^n(t_1^{n,*})| \\ &\geq \frac{c}{b}|t_1^* - t_1^{n,*}| - \|\eta - \eta^n\|_{[0,T_2+r]}. \end{aligned}$$

Hence

$$|t_1^* - t_1^{n,*}| \leq \frac{b}{c}\left(|t_1 - t_1^n| + \|\eta - \eta^n\|_{[0,T_2+r]}\right).$$

This shows $t_1^{n,*} \to t_1^*$, $n \to \infty$, since $t_1^n \to t_1$ by the first part of the proof, and $\|\eta - \eta^n\|_{[0,T_2+r]} \to 0$ by (3.4.6).

The proof for $t_2^{n,*} \to t_2^*$ is analogous. □

The existence of t_2^* allows us to define a return map $P : W_0 \to \mathcal{X} \times Z$ by

$$P(\varphi, \zeta) = \begin{cases} (0, 0) & \text{if } (\varphi, \zeta) = (0, 0), \\ \left(\widehat{v_{t_2^*}}, z(t_2^*)\right) & \text{otherwise,} \end{cases}$$

where $\widehat{v_{t_2^*}} \in \mathcal{X}$ is determined by $\widehat{v_{t_2^*}}(s) = v(t_2^* + s)$ for $s \in [t_2 - t_2^*, 0]$, and $\widehat{v_{t_2^*}}(s) = 0$ for $s \in [-r, t_2 - t_2^*]$.

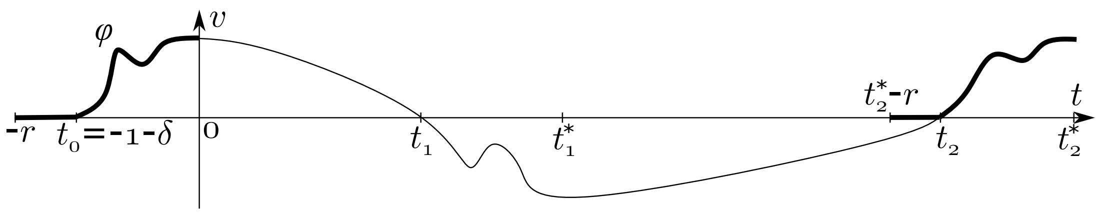


Figure 3.2: The return map P.

Proposition 3.4.7. *P is continuous, and $P(W_0) \subseteq W_0$, $P(W) \subseteq W$.*

Proof. $P(0,0) = (0,0) \in W_0$ trivially. Let $(\varphi,\zeta) \in W$ and $v = v^{\varphi,\zeta}$, $z = z^{\varphi,\zeta}$.

First we prove $P(\varphi,\zeta) = \left(\widehat{v_{t_2^*}}, z(t_2^*)\right) \in W$. By the discussion at the beginning of this section, Theorem 3.3.11 can be applied to get $(v_{t_2^*}, z(t_2^*)) \in \mathcal{X} \times Z$. It is obvious that then $\left(\widehat{v_{t_2^*}}, z(t_2^*)\right) \in \mathcal{X} \times Z$. As $-z(t_2^*) - 1 = t_2 - t_2^*$, it remains to show that

$$[-z(t_2^*) - 1, 0] \ni s \mapsto v\left(t_2^* + s\right) e^{f_1 s} \in \mathbb{R} \text{ is monotone nondecreasing.}$$

If $s \in (-z(t_2^*) - 1, 0]$ then $v(t_2^* + s) > 0$, $v(\eta(t_2^* + s)) < 0$, and

$$\begin{aligned} \frac{d}{ds}\left(v(t_2^*+s)e^{f_1 s}\right) &= \dot{v}(t_2^*+s)e^{f_1 s} + f_1 v(t_2^*+s)e^{f_1 s} \\ &= e^{f_1 s}\left[\left(f_1 - \widetilde{f}(v(t_2^*+s))\right)v(t_2^*+s) - \widetilde{g}(v(\eta(t_2^*+s)))v(\eta(t_2^*+s))\right] > 0. \end{aligned}$$

Thus, $P(\varphi,\zeta) = \left(\widehat{v_{t_2^*}}, z(t_2^*)\right) \in W$ whenever $(\varphi,\zeta) \in W$.

A combination of results in Theorem 3.3.11 and Proposition 3.4.5, 3.4.6 can be used to verify the continuity of P at elements of W.

Continuity of P at $(0,0) \in W_0$ is an easy consequence of Theorem 3.3.11 since for $(\varphi,\zeta) \in W$ and $t_2^* = t_2^*(\varphi,\zeta)$ we have

$$\begin{aligned} \|P(\varphi,\zeta) - P(0,0)\| &= \|\Psi(t_2^*, \varphi + x_*, \zeta) - (x_*, 0) - (0,0)\| \\ &= \|\Psi(t_2^*, \varphi + x_*, \zeta) - \Psi(t_2^*, x_*, 0)\| \\ &\le M\|(\varphi + x_*, \zeta) - (x_*, 0)\| e^{t_2^*(1+L)} \le M\|(\varphi,\zeta)\| e^{(T_2+r)(1+L)}. \end{aligned}$$

□

Let $(\varphi,\zeta) \in \mathcal{X} \times Z$ and $u = u^{\varphi,\zeta}$. Combining the definition of u, $\mathcal{X}$, f_1, using equation (3.4.3) and applying Proposition 3.4.4 we obtain

$$\begin{aligned} \operatorname{lip}(u|_{[-r,T_2+r]}) &\le \max\left\{\operatorname{lip}(u|_{[-r,0]}), \operatorname{lip}(u|_{[0,T_2+r]})\right\} \\ &\le \max\left\{\operatorname{lip}(v_0) + \|v_0\|_{[-r,0]} f_1, c_1\|u\|_{[-r,T_2+r]}\right\} \\ &\le \max\left\{K_1 + f_1 \max\{-A,B\}, c_1 e^{f_1(T_2+r)} \max\{-A,B\}\right\} \\ &\le K_1 + (1+c_1)e^{f_1(T_2+r)} \max\{-A,B\}. \end{aligned}$$

Choose $L_1 > 0$ such that

$$L_1 \ge K_1 + (1+c_1)e^{f_1(T_2+r)} \max\{-A,B\} \quad \text{and} \quad \frac{2cL_1}{c_0 a} \ge \max\{1,-A,B\}.$$

Then, clearly, $\operatorname{lip}(u|_{[-r,T_2+r]}) \le L_1$. Define

$$\beta = \frac{2cL_1}{c_0 a}, \quad \rho = 2^{T_2+r} \quad \text{and} \quad \theta = e^{-(T_2+r)f_1}\beta^{\rho}.$$

Proposition 3.4.8. *For all* $(\varphi,\zeta) \in W$,

$$v(t_2^*) \ge \theta\left(\varphi(0)\right)^{\rho}.$$

Proof. Let $(\varphi,\zeta)\in W$, $u=u^{\varphi,\zeta}$, $\eta=\eta^{\varphi,\zeta}$. Recall that u is monotone decreasing on $[0,t_1^*]$, monotone increasing on $[t_1^*,t_2^*]$, positive on $[0,t_1)\cup(t_2,t_2^*]$, and negative on (t_1,t_2). In addition, $u(\eta(t))<0$ for all $t\in(t_1^*,t_2^*)$.

Define $s_{-1}=t_2^*$ and $s_j=\eta(s_{j-1})$ for $j\in\{0,\dots,k\}$, where k is the unique integer such that $s_k\in(t_1,t_1^*]$. Let

$$m_j=\max_{t\in[s_j,s_{j-1}]}|u(t)|,\qquad j\in\{0,\dots,k\}.$$

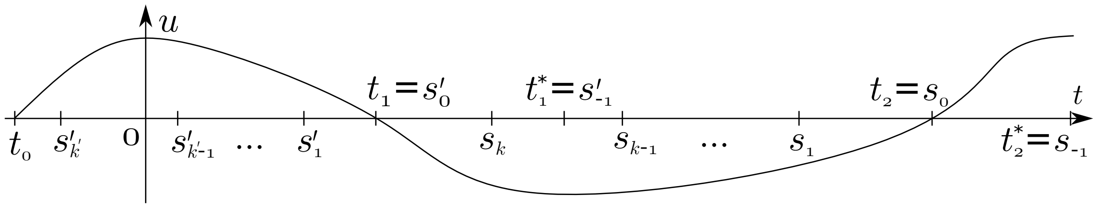


Figure 3.3: The times used in the proof.

Then, by Proposition 3.3.9,

$$\begin{aligned}m_j\ge u(s_{j-1})-u(s_j)&=\int_{s_j}^{s_{j-1}}\dot u(t)\,dt=-\int_{s_j}^{s_{j-1}}C(t)u(\eta(t))\,dt\\&\ge -c_0\int_{s_j}^{s_{j-1}}u(\eta(t))\,dt=-c_0\int_{s_{j+1}}^{s_j}u(t)\,d\eta^{-1}(t)\ge -c_0\frac{a}{c}\int_{s_{j+1}}^{s_j}u(t)\,dt\end{aligned}$$

for $j\in\{0,\dots,k-1\}$. The last integral can be estimated with the area of a rectangular triangle with height m_{j+1} and slope L_1. So, for $j\in\{0,\dots,k-1\}$, we have

$$m_j\ge c_0\frac{a}{c}\int_{s_{j+1}}^{s_j}|u(t)|\,dt\ge c_0\frac{a}{c}\frac{m_{j+1}^2}{2L_1}=\frac{m_{j+1}^2}{\beta}.$$

As u is decreasing on $[t_1,t_1^*]$ and increasing on $[t_1^*,t_2]$, by induction, we have

$$|u(t_2^*)|=m_0\ge\frac{m_k^{2^k}}{\beta^{1+2+\ldots+2^{k-1}}}\ge\left(\frac{m_k}{\beta}\right)^{2^k}=\left(\frac{|u(t_1^*)|}{\beta}\right)^{2^k}.$$

Similarly, define $s'_{-1}=t_1^*$ and $s'_j=\eta(s'_{j-1})$ for $j\in\{0,\dots,k'\}$, where k' is the unique integer such that $s'_k\in(t_0,0]$. Let

$$m'_j=\max_{t\in\left[s'_j,s'_{j-1}\right]}|u(t)|,\qquad j\in\{0,\dots,k'\}.$$

Analogously to the above estimations, we have

$$|u(t_1^*)|=m'_0\ge\frac{m'^{2^{k'}}_{k'}}{\beta^{1+2+\ldots+2^{k'-1}}}\ge\left(\frac{m'_k}{\beta}\right)^{2^{k'}}\ge\left(\frac{|u(0)|}{\beta}\right)^{2^{k'}},$$

and thus

$$|u(t_2^*)| \geq \left(\frac{|u(t_1^*)|}{\beta}\right)^{2^k} \geq \left(\left(\frac{|u(0)|}{\beta}\right)^{2^{k'}}\right)^{2^k} = \left(\frac{|u(0)|}{\beta}\right)^{2^{k+k'}}.$$

We have $k + k' \leq T_2 + r$ and $|u(0)| \leq \beta$ by the choice of β. Using Proposition 3.4.4, our statement follows. □

Let N be the minimal integer with $N \geq T_2 + r$, and define

$$\delta_0 = \frac{de^{-f_1 r}}{2(1+c_1)^N}.$$

Proposition 3.4.9. *If $(\varphi, \zeta) \in \mathcal{X} \times Z$ with $\|\varphi\|_{[-r,0]} \leq \delta_0 e^{f_1 r}$ then $|v(t)| \leq d/2$ for all $t \in [-r, T_2 + r]$.*

Proof. Let $(\varphi, \zeta) \in \mathcal{X} \times Z$ and suppose $\|\varphi\|_{[-r,0]} \leq \delta_0 e^{f_1 r}$.

As $v_0 = \varphi$, Proposition 3.4.4 implies $\|u_0\|_{[-r,0]} \leq \delta_0 e^{f_1 r}$. Assume that, for some $j \in \{0, N-1\}$, we have

$$|u(t)| \leq \delta_0 e^{f_1 r}(1+c_1)^j \quad \text{for all } t \in [-r, j].$$

Then, for $t \in [j, j+1]$, from equation (3.4.3) it can be obtained that

$$\begin{aligned} |u(t)| &\leq \left|u(j) + \int_j^t \dot{u}(s)\, ds\right| \leq |u(j)| + \int_j^{j+1} C(s)|u(s - z(s) - 1)|\, ds \\ &\leq \delta_0 e^{f_1 r}(1+c_1)^j + c_1 \delta_0 e^{f_1 r}(1+c_1)^j = \delta_0 e^{f_1 r}(1+c_1)^{j+1}. \end{aligned}$$

So, by induction, we get $|u(t)| \leq \delta_0 e^{f_1 r}(1+c_1)^N$. Hence, by Proposition 3.4.4 we conclude

$$|v(t)| \leq \delta_0 e^{f_r}(1+c_1)^N = \frac{d}{2}$$

for all $t \in [-r, T_2 + r] \subseteq [-r, N]$. □

Proposition 3.4.10. *If $(\varphi, \zeta) \in W$ with $\varphi(0) \leq \delta_0$, then $z(t_2^*) \leq [\zeta - d/c]^+$.*

Proof. Let $(\varphi, \zeta) \in W$ with $\varphi(0) \leq \delta_0$, and let $v = v^{\varphi,\zeta}$, $z = z^{\varphi,\zeta}$, $y = y^{\varphi,\zeta}$.

From $(\varphi, \zeta) \in W$ it follows that $0 \leq \varphi(s) \leq \varphi(0)e^{f_1 r} \leq \delta_0 e^{f_1 r}$, $s \in [-r, 0]$. Proposition 3.4.9 can be applied to get $|v(t)| \leq d/2$ for all $t \in [-r, T_2 + r]$.

Recall that $t_0 = -\zeta - 1$, $y(t_0) = c\zeta$, and y satisfies equation (3.1.8) a.e. in $[t_0, \infty)$. Moreover, $z(t_2^*) = (1/c)y(t_2^* - z(t_2^*) - 1) = (1/c)y(t_2)$.

Observe that if $y(t) > 0$ on an interval $I \subset [t_0, T_2 + r]$ then, by $|v(t)| \leq d/2$ on $[-r, T_2 + r]$, we have $\dot{y}(t) \leq (d/2) - d = -(d/2)$ a.e. in I. It follows that either $y(t_2) = 0$, or $y(t) > 0$ for all $t \in [t_0, t_2]$. In case $y(t_2) = 0$ the statement trivially holds since $z(t_2^*) =$

$(1/c)y(t_2) = 0$. Assume that $y(t) > 0$ for all $t \in [t_0, t_2]$. Using $t_2 - t_0 \geq t_1^* - t_1 + t_0^* - t_0 \geq 2$, we find

$$z(t_2^*) = \frac{1}{c} y(t_2) = \frac{1}{c} \left(y(t_0) + \int_{t_0}^{t_2} \dot{y}(t)\, dt \right) \leq \frac{1}{c} \left(c\zeta - \frac{d}{2}(t_2 - t_0) \right) \leq \zeta - \frac{d}{c}.$$

This is a contradiction if $\zeta < d/c$ since $z(t_2^*) > 0$.

Therefore, either $\zeta \in [0, d/c]$ and $z(t_2^*) = 0$, or $\zeta > d/c$ and $z(t_2^*) \leq \zeta - d/c$. □

We need a function $\alpha \in C^2([0, q/c], \mathbb{R})$ with the properties

(α1) $\alpha(0) = 0$,

(α2) $\alpha'(\xi) > 0$, $\alpha''(\xi) > 0$ for all $\xi \in (0, q/c]$,

(α3) $\alpha(q/c) \leq \theta(\delta_0)^\rho$,

(α4) $\alpha\big(\xi - (d/c)\big) \leq \theta\big(\alpha(\xi)\big)^\rho$ for all $\xi \in [d/c, q/c]$.

Proposition 3.4.11. *There exists $\alpha \in C^2([0, q/c], \mathbb{R})$ such that (α1)–(α4) are satisfied.*

Proof. We look for α in the form

$$\alpha(\xi) = a_1 \exp\left(-a_2 \exp\left(\frac{a_3}{\xi} \right) \right) \quad \text{for } \xi \in \left(0, \frac{q}{c}\right]$$

with some $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ determined later. For $\xi \in (0, q/c]$, we have

$$\alpha'(\xi) = \frac{a_1 a_2 a_3}{\xi^2} \exp\left(\frac{a_3}{\xi} - a_2 \exp\left(\frac{a_3}{\xi} \right) \right),$$
$$\alpha''(\xi) = \frac{a_1 a_2 a_3}{\xi^4} \left(a_2 a_3 \exp\left(\frac{a_3}{\xi} \right) - a_3 - 2\xi \right) \exp\left(\frac{a_3}{\xi} - a_2 \exp\left(\frac{a_3}{\xi} \right) \right).$$

It is elementary to see that

$$\alpha(\xi) \to 0,\ \alpha'(\xi) \to 0,\ \alpha''(\xi) \to 0 \quad \text{as } \xi \to 0+.$$

Then, by setting $\alpha(0) = 0$, it follows that $\alpha \in C^2([0, q/c], \mathbb{R})$. Condition ($\alpha$1) holds by definition. The property for α' in (α2) is obvious from the above form of $\alpha'(\xi)$. From the above expression for $\alpha''(\xi)$ it is clear that $\alpha''(\xi) > 0$ for all $\xi \in (0, q/c]$ if

$$a_2 a_3 \exp\left(\frac{a_3}{\xi} \right) > a_3 + 2\xi \quad \text{for all } \xi \in \left(0, \frac{q}{c}\right],$$

which is guaranteed by

$$a_2 a_3 \exp\left(\frac{a_3}{(q/c)} \right) > a_3 + 2\frac{q}{c},$$

that is,

$$a_2 > \left(1 + 2\frac{q}{a_3 c}\right) \exp\left(-\frac{a_3 c}{q}\right). \tag{3.4.7}$$

Property (α3) holds if

$$a_1 \leq \theta \delta_0^\rho \exp\left(a_2 \exp\left(\frac{a_3 c}{q}\right)\right). \tag{3.4.8}$$

Inequality (α4) is valid if

$$a_1 \exp\left(-a_2 \exp\left(\frac{a_3}{\xi - (d/c)}\right)\right) \leq \theta a_1^\rho \exp\left(-a_2 \rho \exp\left(\frac{a_3}{\xi}\right)\right)$$

for all $\xi \in (d/c, q/c]$. This inequality holds if both

$$a_1 \leq \theta a_1^\rho \quad \text{and} \quad \exp\left(\frac{a_3}{\xi - (d/c)}\right) \geq \rho \exp\left(\frac{a_3}{\xi}\right) \quad \text{for all } \xi \in \left(\frac{d}{c}, \frac{q}{c}\right]$$

are satisfied, that is, by $\rho > 1$,

$$a_1 \geq \frac{1}{\theta^{\rho - 1}} \tag{3.4.9}$$

and

$$a_3 \geq \xi \left(\frac{c}{d}\xi - 1\right) \log \rho \qquad \text{for all } \xi \in \left(\frac{d}{c}, \frac{q}{c}\right].$$

Since $\xi \mapsto \xi\big((c/d)\xi - 1\big) \log \rho$ is increasing on $(d/c, q/c]$, the last inequality is guaranteed by

$$a_3 \geq \frac{q}{c}\left(\frac{q}{d} - 1\right) \log \rho. \tag{3.4.10}$$

We have to find $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ so that all Inequalities (3.4.7), (3.4.8), (3.4.9) and (3.4.10) are true.

First, fix $a_1 > 0$ so that (3.4.9) is satisfied. Now, choose $a_3^* > 0$ such that (3.4.10) holds for all $a_3 \geq a_3^*$. In the next step, using that the expression on the right hand side of (3.4.7) is monotone decreasing in a_3, we can fix $a_2 > 0$ such that (3.4.7) is valid for all $a_3 \geq a_3^*$. Finally, as a_1 and a_2 are fixed, on can find a sufficiently large $a_3 \geq a_3^*$ so that (3.4.8) holds as well. This completes the proof. □

With the α given in Proposition 3.4.11, recall that $K_0 = (f_1 + g_1)\max\{-A, B\}$, $K_1 = rK_0$, and the sets W_{α,K_0}, W_{α,K_1}, V_{α,K_1} are defined by Formulas (3.1.10), (3.1.11).

Proposition 3.4.12. *The set V_{α,K_1} is a compact and convex subset of $C_{[-1,0]} \times \mathbb{R}$.*

Proof. Compactness of V_{α,K_1} follows in a straighforward way from the definition of V_{α,K_1} and from the Arzela–Ascoli theorem.

In order to show the convexity of V_{α,K_1}, let (ψ^1,ζ^1) and (ψ^2,ζ^2) be in V_{α,K_1}, and set $(\psi,\zeta)=\lambda(\psi^1,\zeta^1)+(1-\lambda)(\psi^2,\zeta^2)$ with some $\lambda\in[0,1]$. Proposition 3.4.11 guarantees the convexity of α. Hence

$$\begin{aligned}\psi(0)&=\lambda\psi^1(0)+(1-\lambda)\psi^2(0)\geq\lambda\alpha\left(\zeta^1\right)+(1-\lambda)\alpha\left(\zeta^2\right)\\&\geq\alpha\left(\lambda\zeta^1+(1-\lambda)\zeta^2\right)=\alpha(\zeta).\end{aligned}$$

All other properties of V_{α,K_1} are obviously preserved by the convex combination. □

It is easy to see that $W_{\alpha,K_1}\subset W_0$. Therefore, the map P is well defined on W_{α,K_1}. We know that W_0 and W are invariant under P. The next result shows the invariance of W_{α,K_1}, and slighlty more since, by $K_0<K_1$, $W_{\alpha,K_0}\subseteq W_{\alpha,K_1}$.

Proposition 3.4.13. $P\left(W_{\alpha,K_1}\right)\subseteq W_{\alpha,K_0}$.

Proof. We have $P(0,0)=(0,0)\in W_{\alpha,K_0}$. Suppose $(\varphi,\zeta)\in W_{\alpha,K_1}\setminus\{(0,0)\}$. Then the inequality $\varphi(0)\geq\alpha(\zeta)$ and the nondecreasing property of $[-r,0]\ni s\mapsto\varphi(s)e^{f_1 s}\in\mathbb{R}$ combined imply that $(\varphi,\zeta)\in W$. By Proposition 3.4.7, $P(\varphi,\zeta)=(\widehat{v_{t_2^*}},z(t_2^*))\in W$. Thus, two facts remain to show: $\operatorname{lip}(\widehat{v_{t_2^*}})\leq K_0$, and that P preserves the property $\varphi(0)\geq\alpha(\zeta)$, i.e., $v(t_2^*)\geq\alpha(z(t_2^*))$.

From equation (3.4.2) and from $v_t\in\mathcal{X}$ it follows that $|\dot v(t)|\leq K_0$ for all $t>0$. Hence the definition of $\widehat{v_{t_2^*}}$ and $0<t_2<t_2^*$ imply $\operatorname{lip}(\widehat{v_{t_2^*}})\leq K_0$.

By $(\varphi,\zeta)\in W_{\alpha,K_1}\setminus\{(0,0)\}\subset W$ we have $\varphi(0)\geq\alpha(\zeta)$, and want to prove $v(t_2^*)\geq\alpha(z(t_2^*))$. There are two cases.

Case 1. $\varphi(0)\geq\delta_0$. Then, by Proposition 3.4.8, properties $(\alpha2)$, $(\alpha3)$ of α, and $z(t_2^*)\in[0,q/c]$, one obtains

$$v(t_2^*)\geq\theta(\varphi(0))^\rho\geq\theta(\delta_0)^\rho\geq\alpha\left(\frac{q}{c}\right)\geq\alpha(z(t_2^*)).$$

Case 2. $\varphi(0)<\delta_0$. Proposition 3.4.10 gives $z(t_2^*)\leq[\zeta-(d/c)]^+$. If $\zeta\leq d/c$ then $z(t_2^*)=0$, and, by $(\alpha1)$, trivially $v(t_2^*)\geq0=\alpha(0)=\alpha(z(t_2^*))$. If $\zeta>d/c$ then applying Proposition 3.4.8, $\varphi(0)\geq\alpha(\zeta)$, $(\alpha4)$ and $(\alpha2)$, we conclude

$$v(t_2^*)\geq\theta(\varphi(0))^\rho\geq\theta(\alpha(\zeta))^\rho\geq\alpha\left(\zeta-\frac{d}{c}\right)=\alpha\left(\left[\zeta-\frac{d}{c}\right]^+\right)\geq\alpha(z(t_2^*)).$$

This completes the proof. □

Define the subsets

$$\begin{aligned}H_1&=\left\{(\psi,\zeta)\in C_{[-1,0]}\times Z\;\middle|\;\psi(-1)=0\right\}\subset C_{[-1,0]}\times\mathbb{R}\\H_r&=\left\{(\varphi,\zeta)\in C_{[-r,0]}\times Z\;\middle|\;\varphi(s)=0\text{ for all }s\in[-r,-\zeta-1]\right\}\subset C_{[-r,0]}\times\mathbb{R}\end{aligned}$$

with the induced subspace topologies.

Introduce the streching map $Q : H_1 \to H_r$ by $Q(\psi, \zeta) = (\varphi, \zeta)$ so that

$$\varphi(s) = \begin{cases} \psi\left(\frac{s}{\zeta+1}\right) & \text{if } s \in [-\zeta - 1, 0], \\ 0 & \text{if } s \in [-r, -\zeta - 1], \end{cases}$$

and the squeezing map $R : H_r \to H_1$ by $R(\varphi, \zeta) = (\psi, \zeta)$ so that

$$\psi(s) = \varphi((\zeta + 1)s) \quad \text{for all } s \in [-1, 0].$$

Proposition 3.4.14. *The maps $Q : H_1 \to H_r$, $R : H_r \to H_1$ are continuous, and*

$$Q\left(V_{\alpha,K_1}\right) \subseteq W_{\alpha,K_1}, \quad R\left(W_{\alpha,K_0}\right) \subseteq V_{\alpha,K_1}.$$

Proof. In order to see the continuity of Q, let $(\psi, \zeta) \in H_1$,

$$(\psi^n, \zeta^n)_{n=0}^{\infty} \in H_1 \quad \text{with} \quad \|(\psi^n, \zeta^n) - (\psi, \zeta)\| \to 0 \quad \text{as } n \to \infty,$$

and let $Q(\psi, \zeta) = (\varphi, \zeta) \in H_r$, $Q(\psi^n, \zeta^n) = (\varphi^n, \zeta^n) \in H_r$, $n \in \mathbb{N}$. By definition, $\varphi(s) = \varphi^n(s)$ for all $s \in [-r, -\max\{\zeta, \zeta^n\} - 1]$. For $s \in [-\min\{\zeta, \zeta^n\} - 1, 0]$ we have

$$\begin{aligned} |\varphi(s) - \varphi^n(s)| &= \left|\psi\left(\frac{s}{\zeta+1}\right) - \psi^n\left(\frac{s}{\zeta^n+1}\right)\right| \\ &\leq \left|\psi\left(\frac{s}{\zeta+1}\right) - \psi\left(\frac{s}{\zeta^n+1}\right)\right| + \left|\psi\left(\frac{s}{\zeta+1}\right) - \psi^n\left(\frac{s}{\zeta^n+1}\right)\right| \\ &\leq \left|\psi\left(\frac{s}{\zeta+1}\right) - \psi\left(\frac{s}{\zeta^n+1}\right)\right| + \|\psi - \psi^n\|_{[-1,0]} \end{aligned}$$

If $s \in [-\max\{\zeta, \zeta^n\} - 1, -\min\{\zeta, \zeta^n\} - 1]$, then in case $\zeta \geq \zeta^n$, one can get

$$|\varphi(s) - \varphi^n(s)| = \left|\psi\left(\frac{s}{\zeta+1}\right) - 0\right| = \left|\psi\left(\frac{s}{\zeta+1}\right) - \psi(-1)\right|,$$

and in case $\zeta < \zeta^n$, we obtain

$$\begin{aligned} |\varphi(s) - \varphi^n(s)| &= \left|0 - \psi^n\left(\frac{s}{\zeta^n+1}\right)\right| \\ &\leq \left|\psi\left(\frac{s}{\zeta^n+1}\right) - \psi^n\left(\frac{s}{\zeta^n+1}\right)\right| + \left|\psi\left(\frac{s}{\zeta^n+1}\right)\right| \\ &\leq \|\psi - \psi^n\|_{[-1,0]} + \left|\psi\left(\frac{s}{\zeta^n+1}\right) - \psi(-1)\right| \end{aligned}$$

For fixed $(\psi, \zeta) \in H_1$, by using the uniform continuity of ψ, the above estimations yield that $\|(\varphi, \zeta) - (\varphi^n, \zeta^n)\|$ tends to zero as n tends to infinity. Since the choice of the sequence (ψ^n, ζ^n) was arbitrary, this shows the continuity of Q at $(\psi, \zeta) \in H_1$. The continuity of R can be obtained analogously.

The inclusion $Q(V_{\alpha,K_1}) \subseteq W_{\alpha,K_1}$ is obvious from the definitions of V_{α,K_1}, W_{α,K_1} and from the fact that the streching does not increase the Lipschitz constant.

Similarly, to prove the inclusion $R(W_{\alpha,K_0}) \subseteq V_{\alpha,K_1}$ we have to check how the squeezing changes the Lipschitz constant and the exponential property. From the definition of R it is clear that the Lipschitz constant of $\psi \in C_{[-1,0]}$, given by $\psi(s) = \varphi((\zeta+1)s)$, $s \in [-1,0]$, can be at most $\zeta + 1 \leq r$ times $\operatorname{lip}(\varphi) \leq K_0$. The facts that

$$[-\zeta-1,0] \ni s \mapsto \varphi(s)e^{f_1 s} \in \mathbb{R} \text{ is nondecreasing} \quad \text{and} \quad r \geq \zeta+1$$

imply that the map

$$[-1,0] \ni s \mapsto \psi(s)e^{f_1 rs} = \varphi((\zeta+1)s)e^{f_1(\zeta+1)s}e^{f_1(r-\zeta-1)s} \text{ is nondecreasing}$$

because it is the product of two nondecreasing functions.

This completes the proof. □

Now we can define a new return map

$$\Pi : V_{\alpha,K_1} \in (\psi,\zeta) \mapsto R \circ P \circ Q(\psi,\zeta) \in V_{\alpha,K_1}.$$

In order to get the ejectivity of the fixed point $(0,0)$, we prove the following proposition.

Proposition 3.4.15. *There exists a constant $\gamma_1 > 0$ with*

$$\sup_{t\geq 0} \left\| v_t^{\varphi,\zeta} \right\| > \gamma_1 \quad \textit{for all } (\varphi,\zeta) \in W. \tag{3.4.11}$$

Proof. Suppose that there is no γ_1 with inequality (3.4.11). Then there exists a sequence $(\varphi^n,\zeta^n)_{n=1}^{\infty}$ in W such that

$$\sup_{t\geq 0} \left\| v_t^{\varphi^n,\zeta^n} \right\|_{[-r,0]} \leq \min\left\{\frac{d}{2}, \frac{1}{n}\right\}.$$

By Proposition 3.4.2, we can assume without loss of generality, that $z^{\varphi^n,\zeta^n}(t) = 0$, $t \geq 0$, $n \in \mathbb{N}$. Setting $v^n = v^{\varphi^n,\zeta^n}$, $n \in \mathbb{N}$, we have

$$\dot{v}^n(t) = -f(v^n(t)) - g(v^n(t-1)) \qquad (t > 0). \tag{3.4.12}$$

Considering the iterates $P^j(\varphi^n,\zeta^n)$, $j \in \mathbb{N}$, $n \in \mathbb{N}$, and taking into account the definition of t_1, t_1^*, t_2, t_2^*, and P, for each $n \in \mathbb{N}$ there is a sequence $(t_k^n)_{k=0}^{\infty}$ such that

$$t_0^n = -1,\ t_k^n + 1 < t_{k+1}^n,\ v^n(t_k^n) = 0, \begin{cases} v^n(t) > 0 & \text{for } t \in \left(t_{2k}^n, t_{2k+1}^n\right), \\ v^n(t) < 0 & \text{for } t \in \left(t_{2k+1}^n, t_{2k+2}^n\right). \end{cases} \tag{3.4.13}$$

for all integers $k \geq 0$.

We claim that

$$\left\|v^n_{t^n_k+1}\right\|_{[-1,0]} \leq e^{f_1} \left|v^n_{t^n_k+1}\right| \quad \text{for all } k \in \mathbb{N} \tag{3.4.14}$$

where f_1 is an upper bound for $\widetilde{f}$.

Recalling functions $\widetilde{f}$, $\widetilde{g}$, equation (3.4.12) can be written in the form (3.4.2) with $z(t) = 0$. Then, for $k \in \mathbb{N}$, by using condition (3.4.13), we obtain

$$\begin{aligned}&\frac{d}{ds}\left[v^n\left(t^n_{2k}+s\right)e^{f_1 s}\right] = \dot{v}^n\left(t^n_{2k}+s\right)e^{f_1 s} + v^n\left(t^n_{2k}+s\right)f_1 e^{f_s}\\
&= \left[\left(f_1 - \widetilde{f}\left(v^n(t^n_{2k}+s)\right)\right)v^n(t^n_{2k}+s) - \widetilde{g}\left(v^n(t^n_{2k}+s-1)\right)v^n(t^n_{2k}+s-1)\right]e^{f_1 s} \geq 0\end{aligned}$$

for all $s \in [0,1]$ because $v^n(t^n_{2k}+s) \geq 0$, $0 \leq \widetilde{f}(v^n(t^n_{2k}+s)) \leq f_1$, $\widetilde{g}(v^n(t^n_{2k}+s-1)) > 0$ and $v^n(t^n_{2k}+s-1) \leq 0$. Thus,

$$0 \leq v^n\left(t^n_{2k}+s\right) \leq v^n\left(t^n_{2k}+1\right)e^{f_1(1-s)} \leq v^n\left(t^n_{2k}+1\right)e^{f_1} \qquad (s \in [0,1]).$$

Analogously, for each nonnegative integer k,

$$0 \geq v^n\left(t^n_{2k+1}+s\right) \geq v^n\left(t^n_{2k+1}+1\right)e^{f_1(1-s)} \geq v^n\left(t^n_{2k+1}+1\right)e^{f_1} \qquad (s \in [0,1]).$$

This proves the claim.

By (S2), (3.4.12) and (3.4.13), we find that $t \mapsto |v^n(t)|$ is a decreasing function on $[t^n_k+1, t^n_{k+1}]$ for all $k \in \mathbb{N}$. This fact, combined with (3.4.14) and the choice of $(\varphi^n, \zeta^n)_{n=1}^{\infty}$, yields, for all $n \in \mathbb{N}$, the existence of an integer $k(n) > n$ such that

$$\frac{1}{2}\sup_{s\geq 0}\left|v^n\left(t^n_{k(n)}+s\right)\right| \leq \left\|v^n_{t^n_{k(n)}+1}\right\| < \frac{1}{n}. \tag{3.4.15}$$

For each $n \in \mathbb{N}$, defining

$$w^n : [-1,\infty) \ni t \mapsto \frac{v^n\left(t^n_{k(n)}+1+t\right)}{\left|v^n\left(t^n_{k(n)+1}\right)\right|},$$

it satisfies $|w^n(0)| = 1$ and, by (3.4.14),

$$\sup_{t\geq -1}|w^n(t)| \leq \frac{1}{\left|v^n\left(t^n_{k(n)}+1\right)\right|}\sup_{s\geq 0}\left|v^n\left(t^n_{k(n)}+s\right)\right| \leq 2\frac{\left\|v^n_{t^n_{k(n)+1}}\right\|_{[-1,0]}}{\left|v^n\left(t^n_{k(n)}+1\right)\right|} \leq 2e^{f_1}.$$

Moreover, equation (3.4.12), the definition of $\widetilde{f}$, $\widetilde{g}$ and w^n imply

$$\dot{w}^n(t) = -\widetilde{f}\left(v^n\left(t^n_{k(n)}+1+t\right)\right)w^n(t) - \widetilde{g}\left(v^n\left(t^n_{k(n)}+t\right)\right)w^n(t-1) \tag{3.4.16}$$

for all $t > 0$. Hence $|\dot{w}^n(t)| \leq 2(f_1+g_1)e^{f_1}$ for all $t > 0$.

We can apply the Arzela–Ascoli theorem and the Cantor diagonalization process for the sequence $\left(w^n|_{[0,\infty)}\right)_{n=1}^{\infty}$ of continuous functions to find a subsequence $(n_l)_{l=1}^{\infty}$ of $\mathbb{N}$ and a continuous function $w : [0,\infty) \to \mathbb{R}$ so that

$$w^{n_l}(t) \to w(t) \text{ as } l \to \infty \text{ uniformly in } t \text{ on compact subsets of } [0,\infty).$$

From (3.4.15) and the definitions of $\widetilde{f}$, $\widetilde{g}$ it follows that

$$\widetilde{f}\left(v^{n_l}\left(t^{n_l}_{k(n_l)}+1+t\right)\right) \to f'(0) \quad \text{and} \quad \widetilde{g}\left(v^{n_l}\left(t^{n_l}_{k(n_l)}+t\right)\right) \to g'(0) \quad \text{as } l \to \infty.$$

Hence the right-hand side of equation (3.4.16) converges to $-f'(0)w(t) - g'(0)w(t-1)$ uniformly on compact subsets of $[1,\infty)$. Consequently, w is differentiable on $(1,\infty)$, and satisfies

$$\dot{w}(t) = -f'(0)w(t) - g'(0)w(t-1) \qquad (t > 1). \tag{3.4.17}$$

So, we obtained a continuous $w : [0,\infty) \to \mathbb{R}$ so that $|w(0)| = 1$, $|w(t)| \leq 2e^{f_1}$ for all $t \geq 0$, the restriction $w|_{(1,\infty)}$ is differentiable and equation (3.4.17) holds. From (3.4.13) observe that w^n has at most one sign change on $[0,1]$, $n \in \mathbb{N}$. Then w can have at most one sign change on $[0,1]$ as well. By Proposition 3.2.1 it follows that w is unbounded on $[0,\infty)$. This is a contradiction, and the proof is complete. □

Proposition 3.4.16. $(0,0) \in V_{\alpha,K_1}$ *is an ejective fixed point of* Π.

Proof. As the maps Q and R act on $(\psi,\zeta) \in C_{[-1,0]} \times \mathbb{R}$ and $(\varphi,\zeta) \in C_{[-r,0]} \times \mathbb{R}$, respectively, such that the norms of ψ and φ are preserved, it suffices to show the ejectivity of the map of the fixed pont $(0,0)$ of P on W_{α,K_1}.

By Propositions 3.4.1, 3.4.15, and by the fact that $(0,0)$ is an equilibrium point, there exists $\gamma_2 > 0$ such that if $(\varphi,\zeta) \in W$ and $\|(\varphi,\zeta)\| = \|\varphi\|_{[-r,0]} + \zeta < \gamma_2$ then

$$\left\|\left(v_t^{\varphi,\zeta}, z^{\varphi,\zeta}(t)\right)\right\| = \left\|v_t^{\varphi,\zeta}\right\|_{[-r,0]} + z^{\varphi,\zeta}(t) < \gamma_1 \text{ for all } t \in [0, T_2 + r].$$

Indirectly, suppose that there exists $(\varphi,\zeta) \in W$ so that

$$\left\|P^k(\varphi,\zeta)\right\| < \gamma_2 \quad \text{for all } k \in \{0,1,2,\ldots\}. \tag{3.4.18}$$

For $k \in \{3,4,\ldots\}$, define $t_k = \min\{t > t_{k-1} \mid v(t) = 0\}$. Observe that, by the choice of γ_2, for each fixed $k \in \{0,1,2,\ldots\}$, the inequality $\|P^k(\varphi,\zeta)\| < \gamma_2$ and the fact that the solutions of system (3.1.7), (3.1.8), (3.1.9) generate a semiflow imply that

$$\left\|\left(v_t^{\varphi,\zeta}, z^{\varphi,\zeta}(t)\right)\right\| = \left\|v_t^{\varphi,\zeta}\right\|_{[-r,0]} + z^{\varphi,\zeta}(t) < \gamma_1 \quad \text{for all } t \in [t_k, t_k + T_2 + r].$$

Recall that $t_2^* \leq T_2 + r$ in the definition of P. Thus, from (3.4.18), it can be obtained by induction that

$$\left\|v_t^{\varphi,\zeta}\right\|_{[-r,0]} \leq \left\|v_t^{\varphi,\zeta}\right\|_{[-r,0]} + z^{\varphi,\zeta}(t) < \gamma_1 \quad \text{for all } t \geq 0.$$

This inequality contradicts the existence of $\gamma_1 > 0$ with inequality (3.4.11).

Therefore, ejectivity of the trivial fixed point $(0,0)$ of P on W_{α,K_1} follows with the open set $W_{\alpha,K_1} \cap U$, where

$$U = \{(\varphi,\zeta) \in C_{[-r,0]} \times \mathbb{R} : \ \|(\varphi,\zeta)\| < \gamma_2\}.$$

The proof is complete. □

Now we are able to show the main result.

Theorem 3.4.17. *Assume that Conditions (S1)–(S4) hold. Then system* (3.1.7)*,* (3.1.8)*,* (3.1.9) *has a slowly oscillatory periodic solution.*

Proof. By Proposition 3.4.12 the set V_{α,K_1} is a compact and convex subset subset of the Banach space $C_{[-1,0]} \times \mathbb{R}$. Propositions 3.4.13, 3.4.7, 3.4.14 combined show that the map $\Pi : V_{\alpha,K_1} \to V_{\alpha,K_1}$ is continuous. According to Proposition 3.4.16 the fixed point $(0,0)$ of Π is ejective. Then Theorem C guarantees that Π has a nonejective fixed point (ψ^*,ζ^*) in V_{α,K_1}. By the ejectivity of $(0,0)$, we have $(\psi^*,\zeta^*) \neq (0,0)$, in particular $\psi^* \neq 0$.

Define $\varphi^* \in C_{[-r,0]}$ so that $(\varphi^*,\zeta^*) = Q(\psi^*,\zeta^*)$. Let $(\varphi^{**},\zeta^{**}) = P(\varphi^*,\zeta^*)$. From $R(\varphi^{**},\zeta^{**}) = (\psi^*,\zeta^*)$ one obtains $\zeta^{**} = \zeta^*$. Therefore, $\varphi^{**}(s) = 0 = \varphi^*(s)$ for all $s \in [-r, -\zeta^* - 1]$. Moreover, Q streches ψ^* with the same factor $\zeta^* + 1$ as R squeezes φ^{**}. Then necessarily

$$\varphi^*(s) = \psi^*\left(\frac{s}{\zeta^*+1}\right) = \varphi^{**}\left((\zeta^*+1)\frac{s}{\zeta^*+1}\right) = \varphi^{**}(s)$$

for all $s \in [-\zeta^* - 1, 0]$. Therefore, $(\varphi^{**},\zeta^{**}) = (\varphi^*,\zeta^*)$, that is, $(\varphi^*,\zeta^*) = Q(\psi^*,\zeta^*)$ is a nontrivial fixed point of P.

The solution $(v^{\varphi^*,\zeta^*}, z^{\varphi^*,\zeta^*})$ of system (3.1.7), (3.1.8), (3.1.9) defines a slowly oscillatory periodic solution $(v,z) : \mathbb{R} \to \mathbb{R}$ in the following way. As (φ^*,ζ^*) is a fixed point of P, the restriction $v^{\varphi^*,\zeta^*}|_{[0,\infty)}$ of v^{φ^*,ζ^*} and z^{φ^*,ζ^*} are t_2^*-periodic functions with $t_2^* = t_2^*(\varphi^*,\zeta^*) > 0$. A t_2^*-periodic extension of $v^{\varphi^*,\zeta^*}|_{[0,\infty)}$ and z^{φ^*,ζ^*} from $[0,\infty)$ to $\mathbb{R}$ give the slowly oscillating periodic solution $(v,z) : \mathbb{R} \to \mathbb{R}$. □

3.5 Examples

1. Consider system (3.1.4), (3.1.2), (3.1.3) with $U \in C^2((0,\infty),\mathbb{R})$ and $p \in C^1((0,\infty),\mathbb{R})$ satisfying

$$U'(\xi) > 0, \ U''(\xi) < 0 \ p(\xi) > 0, \ p'(\xi) > 0 \quad \text{for all } \xi \geq 0.$$

Then $U'' - p' < 0$, so $U' - p$ has at most one zero. Assume that there exists $x_* \in (0,c)$ with $U'(x_*) - p(x_*) = 0$. Then x_* is the optimal rate.

For fixed constants $\kappa, a, b, q, r_0, r_1$ with $\kappa > 0$, $0 < a < x_* < c < b$, $q > 0$, $r_0 \geq 0$, $r_1 > 0$ set $K = \kappa[\max_{\xi\in[a,b]} \xi U'(\xi) + \max_{\xi\in[a,b]} \xi p(\xi)]$. Define X, Y, Z and $G : X \times Z \to \mathbb{R}$ as in Section 3.1. Then, for $F(\varphi, \psi) = G(\varphi, \sigma(\psi))$, $(\varphi, \psi) \in X \times Y$, Hypothesis (H2) holds. The Lipschitz continuity in (H1) can be obtained easily from the smoothness of U, p and the Lipschitz properties for X, Y, σ. (H3) is valid with $r_2 = r_1$ by the definition of σ. Hypothesis (H4) requires the additional condition

$$aU'(a) > \max_{\xi\in[a,b]} \xi p(\xi), \quad bU'(b) < \min_{\xi\in[a,b]} \xi p(\xi). \tag{3.5.1}$$

Under the above assumptions, Theorems 3.3.5, 3.3.11 yield that system (3.1.4), (3.1.2), (3.1.3) is well posed both in $X \times Y$ and $X \times Z$, and all solutions can be extended to the right half line.

2. In system (3.1.4), (3.1.2), (3.1.3) choose $r_0 = 0$, $r_1 = 1$, and $U(\xi) = -\xi^{-\alpha}/\alpha$, $p(\xi) = \xi^\beta$ with some positive α and β. Then $U'(\xi) = \xi^{-\alpha-1}$, $\xi U'(\xi) = \xi^{-\alpha}$, $\xi p(\xi) = \xi^{\beta+1}$, and $x_* = 1$. It is straightforward that with a fixed $c > 1$ all conditions of Section 3.1 are satisfied provided there are constants a, b so that $0 < a < 1 < c < b$ and condition (3.5.1) holds. In our particular case condition (3.5.1) holds if $a^{-\alpha} > b^{\beta+1}$ and $b^{-\alpha} < a^{\beta+1}$, or equivalently $a^\alpha b^{\beta+1} < 1 < a^{\beta+1} b^\alpha$. This can be true only if $\beta + 1 < \alpha$, and even with $\beta + 1 < \alpha$ we cannot choose $a > 0$ arbitrarily small, $b > 1$ arbitrarily large.

In order to satisfy condition (3.5.1) we modify function U close to zero. For $\varepsilon \in (0, 1)$ define

$$U_\varepsilon(\xi) = -\frac{1}{\alpha\xi^\alpha} - V_\varepsilon(\xi), \quad \text{where} \quad V_\varepsilon(\xi) = \begin{cases} \exp\left(\frac{1}{\xi} + \frac{1}{\xi-\varepsilon}\right) & \text{if } 0 < \xi < \varepsilon, \\ 0 & \text{if } \xi \geq \varepsilon. \end{cases}$$

Cleary, V_ε and U_ε are in $C^\infty((0, \infty), \mathbb{R})$, and $\xi U'_\varepsilon(\xi) = \xi^{-\alpha} + [\xi^{-1} + \xi/(\xi - \varepsilon)^2]\, V_\varepsilon(\xi)$ for all $\xi > 0$. We want to find a, b such that $0 < a < 1 < b$, and $aU'_\varepsilon(a) > b^{\beta+1}$ and $b^{-\alpha} < a^{\beta+1}$. For given $a > 0$ choose $b > 0$ such that $b^{-\alpha} = a^{\beta+1}/2$, i.e., $b = 2^{1/\alpha} a^{-(\beta+1)/\alpha}$. Then $b^{-\alpha} < a^{\beta+1}$ holds. Inequality $aU'_\varepsilon(a) > b^{\beta+1}$ is satisfied if

$$aU'_\varepsilon(a) > 2^{\frac{\beta+1}{\alpha}} a^{-\frac{(\beta+1)^2}{\alpha}},$$

which is valid if $a > 0$ is small enough since $aU'_\varepsilon(a) \to \infty$ faster than $a^{-(\beta+1)^2/\alpha}$ as $a \to 0^+$. Consequently, for each fixed $\varepsilon \in (0, 1)$, there exists $a = a_\varepsilon \in (0, \varepsilon)$ so that, by choosing $a \in (0, a_\varepsilon)$ and $b = 2^{1/\alpha} a^{-(\beta+1)/\alpha}$, condition (3.5.1) is valid with U_ε instead of U. Clearly, $b \to \infty$ as $a \to 0^+$. In particular, we may assume that $b > c$).

Therefore, for each $\varepsilon \in (0, 1)$, Theorem 3.3.11 is applicable for system (3.1.4), (3.1.2), (3.1.3) with $r_0 = 0$, $r_1 = 1$, $p(\xi) = \xi^\beta$ and U_ε instead of U. For the new variable $v = x - 1$ we obtain system (3.1.7), (3.1.8), (3.1.9) with $f(v) = -\kappa\,[(v + 1)U'_\varepsilon(v + 1) - U'(1)]$, $g(v) = \kappa\left[(v + 1)^{\beta+1} - 1\right]$, and $d = c - 1 > 0$.

It is easy to see that Conditions (S1)–(S3) hold with $A = a - 1$, $B = b - 1$. We have $f'(0) = \kappa\alpha$ and $g'(0) = \kappa(\beta + 1)$.

If $\alpha > \beta + 1$ then (S5) holds. Indeed, let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$, and suppose $\lambda + \kappa\alpha + \kappa(\beta+1)e^{-\lambda} = 0$. Then $\kappa\alpha \leq |\lambda + \kappa\alpha| = |\kappa(\beta+1)e^{-\lambda}| \leq \kappa(\beta+1)$, a contradiction to $\alpha > \beta + 1$. Therefore, by Theorem 3.4.3, the $(0,0)$ solution of system (3.1.7), (3.1.8), (3.1.9) is locally asymptotically stable.

Assume $\alpha < \beta+1$. Then there exists $\vartheta_0 \in (\pi/2, \pi)$ so that $-\cos\vartheta_0 = \alpha/(\beta+1)$. Define $\kappa_0 = -(1/\alpha)\vartheta_0 \cot\vartheta_0$. For each $\kappa > \kappa_0$ there exists $\vartheta_1 \in (\vartheta_0, \pi)$ such that $\kappa\alpha = -\vartheta_1 \cot\vartheta_1$ since $[\pi/2, \pi) \ni \vartheta \mapsto -\vartheta\cot\vartheta \in \mathbb{R}$ increases from 0 to ∞. Then

$$\kappa(\beta+1) = \frac{\beta+1}{\alpha}\kappa\alpha = -\frac{1}{\cos\vartheta_0}\left(-\vartheta_1\cot\vartheta_1\right) = \frac{-\cos\vartheta_1}{-\cos\vartheta_0}\frac{\vartheta_1}{\sin\vartheta_1} > \frac{\vartheta_1}{\sin\vartheta_1},$$

and condition (3.2.2) is satisfied implying (S4) for all $\kappa > \kappa_0$. Thus, Theorem 3.4.17 implies that, with the above particular choice of f, g, system (3.1.7), (3.1.8), (3.1.9) has a slowly oscillatory periodic solution provided $\kappa > \kappa_0$ and $\alpha < \beta + 1$. Equivalently, if $\alpha < \beta + 1$ and $\kappa > \kappa_0$ then system (3.1.4), (3.1.2), (3.1.3) with $r_0 = 0$, $r_1 = 1$, $p(\xi) = \xi^\beta$ and U_ε instead of U has a periodic solution (x, z) oscillating slowly around $x_* = 1$. For this periodic solution x, we claim that

$$x(t) \in \left[(1+\kappa r)^{-\frac{\beta+1}{\alpha}}, 1+\kappa r\right] \quad \text{for all } t \in \mathbb{R}. \tag{3.5.2}$$

Let $t_1 \geq 0$ be such that $x(t_1) > 1$ and x has a local maximum at t_1. Then $\dot{x}(t_1) = 0$. If $x(t) > 1$ for all $t \in [t_1 - r, t_1]$ then, by $x(t_1)U_\varepsilon'(x(t_1)) < 1$ and $x(t_1 - z(t_1) - 1) > 1$, one obtains

$$\dot{x}(t_1) = \kappa\left[x(t_1)U_\varepsilon'(x(t_1)) - [x(t_1 - z(t_1) - 1)]^{\beta+1}\right] < 0,$$

a contradiction. Therefore, there is a maximal $t_0 \in [t_1 - r, t_1)$ such that $x(t_0) = 1$. An integration gives

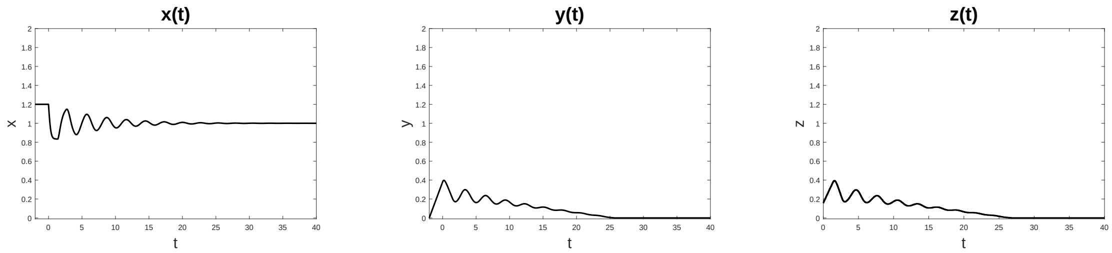
$$x(t_1) = 1 + \int_{t_0}^{t_1} \kappa\left[x(t)U_\varepsilon'(x(t)) - [x(t - z(t_1) - 1)]^{\beta+1}\right] dt \leq 1 + \kappa r,$$

the upper bound in (3.5.2). If $t_2 \in \mathbb{R}$ is such that $x(t_2) < 1$ and x has a local minimum at t_2, then $\dot{x}(t_2) = 0$ and $x(t_2)U_\varepsilon'(x(t_2)) = [x(t_2 - z(t_2) - 1)]^{\beta+1}$. Hence, using $U_\varepsilon \geq U$, the inequality

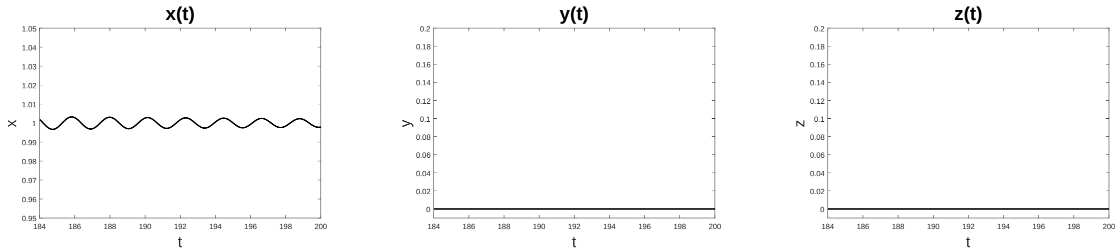
$$[x(t)]^{-\alpha} = x(t)U'(x(t)) \leq x(t)U_\varepsilon'(x(t)) \leq [1+\kappa r]^{\beta+1} \quad \text{for all } t \in \mathbb{R}$$

follows, yielding the lower bound in (3.5.2).

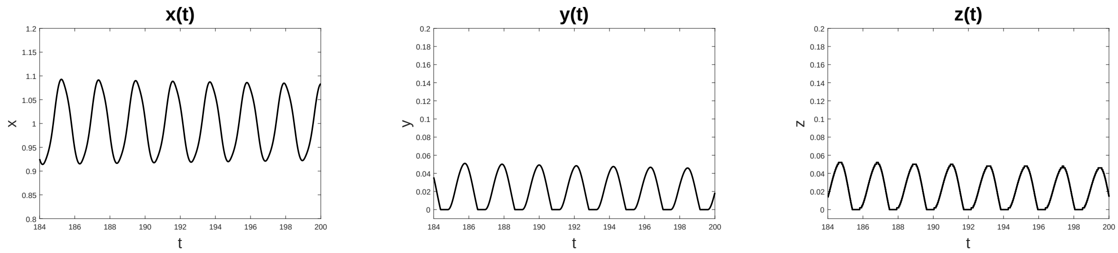
Consequently, if, for a fixed $\kappa > 0$, we choose $\varepsilon > 0$ so that $\varepsilon < (1+\kappa r)^{-(\beta+1)/\alpha}$, and a, b such that condition (3.5.1) and $a \in (0, \varepsilon)$, $b > \max\{c, 1+\kappa r\}$ are satisfied, then all possible periodic solutions (oscillating around $x_* = 1$) of system (3.1.4), (3.1.2), (3.1.3), with $r_0 = 0$, $r_1 = 1$, $p(\xi) = \xi^\beta$ and U_ε instead of U, satisfy the system with the original U as well.



(a) For $\kappa = 1$, the solution tends to the globally asymptotically stable equilibrium, the queue disappears and the delay becomes constant.



(b) For $\kappa = 4$, the solution is asymptotically periodic, there is no queue, so the delay is constant.



(c) For $\kappa = 10$, the solution is asymptotically periodic, the length of the queue and the waiting time are not identically zero. In this case the delay of our system is state-dependent indeed.

Figure 3.4: Numerical solutions with $\alpha = 3$, $\beta = 1$, $q = C = 1.01$, $x_* = 1$, $r_0 = 0$, $r_1 = 1$.

Summary

The thesis summarizes the results of Balázs and Krisztin [4, 3]. It has three chapthers.

Chapter 1 is the introduction, it shows the sketch of the thesis.

The aim of Chapter 2 is to prove the global stability conjecture for the price model of Erdélyi, Brunovský and Walther [9, 8, 37],

$$\dot{x}(t) = a[x(t) - x(t-1)] - \beta|x(t)|x(t). \tag{2.1.1}$$

Garab, Kovács and Krisztin [14] obtained global asymptotic stability of $x = 0$ for equation (2.1.1) provided $a \in (0, 0.61)$. The technique of [14] worked for the more general price model

$$\dot{x}(t) = a\sum_{i=1}^{n} b_i[x(t-s_i) - x(t-r_i)] - g(x(t)). \tag{2.1.2}$$

[14] proved global asymptotic stability for equation (2.1.2) when $a \in (0, 1)$ and an additional condition was assumed, but it remained open to prove global asymptotic stability without the additional condition, i.e., for $a \in (0, 1)$.

By using Stieltjes integrals, equations (2.1.1) and (2.1.2) can be written as

$$\dot{x}(t) = a\int_0^r x(t-s)d\eta(s) - g(x(t)), \tag{2.1.3}$$

$$\dot{y}(t) = a\int_0^r \dot{y}(t-s)d\mu(s) - g(y(t)), \tag{2.1.5}$$

assuming Hypotheses ($\mathrm{H_g}$), (H_η) and (H_μ).

In Section 2.3, we consider equation (2.1.5), formulate the hypotheses on μ, and introduce a suitable phase space. First it is shown that all solutions can be globally extended to $[-r, \infty)$. In Theorem 2.3.2 a sufficient condition is given for the global asymptotic stability of the zero solution of equation (2.1.5). The proof is based on a Lyapunov functional which has been inspired by the one employed for the equation

$$\dot{x}(t) = a\dot{x}(t-1) - g(x(t)) \tag{2.1.6}$$

in the book of Kolmanovskii and Myshkis [21, Chapter 9, p. 374].

In Section 2.4, we consider equation (2.1.3) under Hypotheses ($\mathrm{H_g}$) and (H_η). Combining the global stability result of Section 2.3 for equation (2.1.5) and the continuous

dependence on initial data for equation (2.1.3), the main result, stated in Theorem 2.4.2, is that the zero solution of equation (2.1.3) is globally asymptotically stable provided $a \in (0, 1)$. As a consequence, global asymptotic stability is obtained for the zero solution of the Erdélyi–Brunovský–Walther equation (2.1.1) and also for equation (2.1.2) for the full conjectured region $a \in (0, 1)$, see Corollaries 2.4.3, 2.4.4.

Finally in Section 2.5 we show that the global stability result for equation (2.1.3) is optimal in the sense that for $a > 1$ under the additional condition $g'(0) = 0$ the zero solution is unstable. In addition, some open problems are mentioned.

In Chapter 3 we consider a network model that was introduced by Ranjan, La and Abed in [31, 30]. It contains a single user and a single server. The user sends data by rate $x(t)$ to the server for procession. The server processes the incoming data by the capacity c. Kelly [19] introduced the utility $U(x)$ and the price $p(x)$ per unit flow of the procession, and proposed an end user rate control algorithm as a differential equation.

As the rate $x(t)$ can be larger than the capacity of the server, the data arriving at the server may form a single waiting line (a queue) with length $y(t)$ before procession. Suppose that a unit of data, whose procession was completed and the user received an acknowledgement about it at time t, arrived at the queue $\tau(t)$ time earlier, found a queue with length $y(t-\tau(t))$, and spent waiting time $z(t) = (1/c)y(t-\tau(t))$ in the queue before its procession started. Then the model can be described by the system of equations

$$\dot{x}(t) = \kappa\big[x(t)U'(x(t)) - x(t - r_0 - z(t) - r_1)p(x(t - z(t) - r_1))\big], \tag{3.1.4}$$

$$\dot{y}(t) = \begin{cases} x(t - r_0) - c & \text{if } 0 < y(t) < q, \\ [x(t - r_0) - c]^+ & \text{if } y(t) = 0, \\ -[x(t - r_0) - c]^- & \text{if } y(t) = q, \end{cases} \tag{3.1.2}$$

$$z(t) = \frac{1}{c}y(t - z(t) - r_1). \tag{3.1.3}$$

First we consider a slightly more general system of equations

$$\dot{x}(t) = F(x_t, y_t) \tag{3.1.5}$$

and (3.1.2) in $X \times Y$. The phase space $X \times Y$ contains all possible segments (x_t, y_t).

In order to see that system (3.1.4), (3.1.2), (3.1.3) is a particular case of system (3.1.5), (3.1.2) introduce $Z = [0, q/c] \subset \mathbb{R}$ as a state space for the variable $z(t)$. A cruical fact is the existence of a unique Lipschitz continuous map $\sigma : Y \to Z$ such that

$$\sigma(\psi) = \frac{1}{c}\psi(-\sigma(\psi) - r_1) \qquad (\psi \in Y).$$

Then, for a solution $(x, y) : [-r, \infty) \to \mathbb{R}^2$ of system (3.1.5), (3.1.2) in the phase space $X \times Y$, defining $z(t) = \sigma(y_t)$, $t \geq 0$, equation (3.1.3) is always satisfied for all $t \geq 0$.

Assume that a map $G : X \times Z \to \mathbb{R}$ is given such that, with the particular choice

$$F : X \times Y \ni (\varphi, \psi) \mapsto G(\varphi, \sigma(\psi)) \in \mathbb{R},$$

Hypotheses (H1)–(H4) hold. In this case system (3.1.5), (3.1.2) is equivalent to the system composed of the equations

$$\dot{x}(t) = G(x_t, z(t)), \tag{3.1.6}$$

(3.1.2) and (3.1.3). Then, in the phase space $X \times Y$, for each $(\varphi, \psi) \in X \times Y$, system (3.1.6), (3.1.2), (3.1.3) has the unique solution $x^{\varphi,\psi}[-r, \infty) \to \mathbb{R}$, $y^{\varphi,\psi} : [-r, \infty) \to \mathbb{R}$, $z^{\varphi,\psi} : [0, \infty) \to \mathbb{R}$ where $\left(x^{\varphi,\psi}, y^{\varphi,\psi}\right)$ is the solution of system (3.1.5), (3.1.2), and $z^{\varphi,\psi}(t) = \sigma(y_t^{\varphi,\psi})$, $t \geq 0$.

In Section 3.3 we show that, under Hypotheses (H1)–(H4), for each $(\varphi, \psi) \in X \times Y$, system (3.1.5), (3.1.2) has a unique maximal solution $\left(x^{\varphi,\psi}, y^{\varphi,\psi}\right) : [-r, \infty) \to \mathbb{R}^2$, see Theorem 3.3.5. The solutions define the continuous semiflow

$$\Phi : [0, \infty) \times X \times Y \ni (t, \varphi, \psi) \mapsto \left(x_t^{\varphi,\psi}, y_t^{\varphi,\psi}\right) \in X \times Y,$$

and, for each $t \geq 0$, the solution operators $\Phi(t, \cdot, \cdot) : X \times Y \to X \times Y$ are Lipschitz continuous.

In Theorem 3.3.11, we also show that system (3.1.6), (3.1.2), (3.1.3) can be studied not only in the phase space $X \times Y$, but also in $X \times Z$ with a different notion of solution. The key technical result is that there is a unique Lipschitz continuous map $\gamma : X \times Z \to Y$ so that $\psi = \gamma(\varphi, \zeta)$ satisfies $\psi(s) = c\zeta$ for $s \in [-r, -\zeta - r_1]$, and equation (3.1.2) holds a.e. in $[-\zeta - r_1, 0]$. In particular, $\zeta = (1/c)\psi(-\zeta - r_1)$. This means that the past of the length of the queue can be recovered from the past of the rate (that is $\varphi \in X$) and from the present waiting time. The maps h and k between the two different phase spaces are Lipschitz continuous, h is injective, but k is not, $k \circ h = \mathrm{id}_{X \times Z}$, and $h \circ k\big|_{h(X \times Z)} = \mathrm{id}_{h(X \times Z)}$. Then, for each $(\varphi, \zeta) \in X \times Z$, there exists a unique solution $x^{\varphi,\zeta} : [-r, \infty) \to \mathbb{R}$, $z^{\varphi,\zeta} : [0, \infty) \to \mathbb{R}$ of system (3.1.6), (3.1.2), (3.1.3) in the phase space $X \times Z$ satisfying the initial condition $x_0^{\varphi,\zeta} = \varphi$, $z^{\varphi,\zeta}(0) = \zeta$. Moreover,

$$\Psi : [0, \infty) \times X \times Z \ni (t, \varphi, \zeta) \mapsto \left(x_t^{\varphi,\zeta}, z^{\varphi,\zeta}(t)\right) \in X \times Z$$

is a continuous semiflow on $X \times Z$, and $\Psi(t, \varphi, \zeta) = k(\Phi(t, h(\varphi, \zeta)))$ for all $t \geq 0$.

In Section 3.4 we assume $r_0 = 0$, $r_1 = 1$ and consider system (3.1.4), (3.1.2), (3.1.3). Condition $r_0 = 0$ guarantees a single delay in equation (3.1.4), $r_1 = 1$ can be achieved by rescaling the time. Then for the new variable $v = x - x_*$ we can rewrite our system. Theorem 3.3.11 implies that system (3.1.7), (3.1.8), (3.1.9) is well posed in the phase space $\mathcal{X} \times Z$.

A solution (v, z) of system (3.1.7), (3.1.8), (3.1.9) is called *slowly oscillatory* if for any two zeros t_1, t_2 of v with $t_1 < t_2$ the inequality $z(t_2) + 1 < t_2 - t_1$ holds. This means that the distance between consecutive zeros of v is larger than the delay.

We introduce the sets W and $W_0 = W \cup \{(0,0)\}$. Then, for each $(\varphi, \zeta) \in W$, the solution $v = v^{\varphi,\zeta} : [-r, \infty) \to \mathbb{R}$, $z = z^{\varphi,\zeta} : [0, \infty) \to \mathbb{R}$ is slowly oscillatory with infinite number of zeros. The second zero t_2 of v in $(0, \infty)$ determines $t_2^* > t_2$ so that $t_2 = t_2^* - z(t_2^*) - 1$, and a return map $P : W_0 \to W_0$ can be defined. A nontrivial fixed point of P corresponds to a slowly oscillating periodic solution. A classical tool, that we apply here as well, is Browder's non-ejective fixed point theorem. A large part of Section 3.4 is devoted to the construction of a suitable subset of $\mathcal{X} \times Z$ where Browder's theorem is applicable.

It is a crucial result that $P(\varphi, \zeta)$ cannot decay too fast: there are constants $\theta > 0$, $\rho > 0$ with $v^{\varphi,\zeta}(t_2^*) \geq \theta\,(\varphi(0))^\rho$ for all $(\varphi, \zeta) \in W$. This fact allows to construct a proper C^2-function α. Defining the compact subsets W_{α,K_1} and W_{α,K_0} of $\mathcal{X} \times Z$, the inclusion $P(W_{\alpha,K_1}) \subseteq W_{\alpha,K_0}$ is satisfied. However, W_{α,K_1} and W_{α,K_0} are not convex. Following [25], the subset V_{α,K_1} of $C_{[-1,0]} \times \mathbb{R}$ is compact and convex. Set V_{α,K_1} can be mapped into W_{α,K_1} by the streching map Q given by $Q(\psi, \zeta) = (\varphi, \zeta)$ with $\varphi(s) = \psi(s/(\zeta + 1))$, $s \in [-\zeta - 1, 0]$, and $\varphi\big|_{[-r,-\zeta-1]} \equiv 0$. The squeezing map R, defined by $R(\varphi, \zeta) = (\psi, \zeta)$ with $\psi(s) = \varphi((\zeta + 1)s)$, $s \in [-1, 0]$, maps W_{α,K_0} into V_{α,K_1}. Browder's theorem can be applied for finding a non-ejective fixed point of the map $\Pi = R \circ P \circ Q$ in V_{α,K_1}. This yields a non-ejective fixed point of P in W_{α,K_1} as well. The non-ejective fixed point is nontrivial provided $(0, 0) \in W_{\alpha,K_1}$ is ejective. Ejectivity of $(0, 0) \in W_{\alpha,K_1}$ follows in a standard way from that of the zero solution of the constant delay equation $\dot{v}(t) = -f(v(t)) - g(v(t-1))$. So we can state our main result in Theorem 3.4.17.

Finally, Section 3.5 gives examples.

Összefoglaló

A disszertáció összefoglalja Balázs és Krisztin [4, 3] eredményeit. Három fejezete van.

Az 1. Fejezet a bevezetés, ez bemutatja a disszertáció vázlatát.

A 2. Fejezet célja bebizonyítani Erdélyi, Brunovský és Walther [9, 8, 37] globális stabilitásra vonatkozó sejtését az

$$\dot{x}(t) = a[x(t) - x(t-1)] - \beta|x(t)|x(t) \tag{2.1.1}$$

ármodellre. Garab, Kovács és Krisztin [14] az $x = 0$ globális stabilitását mutatta meg az (2.1.1) egyenletre, feltéve, hogy $a \in (0, 0.61)$. [14] technikája az általánosabb

$$\dot{x}(t) = a \sum_{i=1}^{n} b_i[x(t-s_i) - x(t-r_i)] - g(x(t)). \tag{2.1.2}$$

ármodellre is működött. [14] globális aszimptotikus stabilitást bizonyított az (2.1.2) egyenletre, ha $a \in (0, 1)$, és egy további feltételt teszünk, de nyitott maradt a globális stabilitás bizonyítása ezen plusz feltétel nélkül, azaz $a \in (0, 1)$-re.

Stieltjes-integrálokat használva, a (2.1.1) és (2.1.2) egyenletek

$$\dot{x}(t) = a \int_0^r x(t-s)d\eta(s) - g(x(t)), \tag{2.1.3}$$

$$\dot{y}(t) = a \int_0^r \dot{y}(t-s)d\mu(s) - g(y(t)), \tag{2.1.5}$$

alakban írhatók, feltéve a $(\mathrm{H_g})$, (H_η) és (H_μ) Hipotéziseket.

A 2.3. Szakaszban a (2.1.5) egyenletet tekintjük, megfogalmazzuk a hipotéziseket μ-re, és bevezetjük a megfelelő fázisteret. Először megmutatjuk, hogy minden megoldás globálisan kiterjeszthető $[-r, \infty)$-re. A 2.3.2 Tételben elégséges feltételt adunk a (2.1.5) zéró megoldásának globális aszimptotikus stabilitására. A bizonyítás egy olyan Ljapunov-funkcionálon alapszik, amilyet Kolmanovskii és Myshkis könyve, [21, 9. Fejezet, 374. oldal] alkalmaz az

$$\dot{x}(t) = a\dot{x}(t-1) - g(x(t)) \tag{2.1.6}$$

egyenletre.

A 2.4. Szakaszban a (2.1.3) egyenletet tekintjük a $(\mathrm{H_g})$ és (H_η) hipotézisek mellett. A 2.3. Szakasz (2.1.5) egyenletre vonatkozó globális stabilitási eredményét és a (2.1.3) egyenlet megoldásainak kezdeti értékétől vett folytonos függését kombinálva adódik a fő eredmény, melyet a 2.4.2 Tételben mondunk ki, hogy a (2.1.3) egyenlet zéró megoldása globálisan aszimptotikusan stabil, feltéve, hogy $a \in (0,1)$. Következésképpen kapjuk a zéró megoldás globális aszimptotikus stabilitását Erdélyi–Brunovský–Walther (2.1.1) egyenletében és a (2.1.2) egyenletben az $a \in (0,1)$ paraméterre, amire a sejtés vonatkozott, lásd a 2.4.3, 2.4.4 Következményeket.

Végül, a 2.5. Szakaszban megmutatjuk, hogy a (2.1.3) egyenletre kapott globális stabilitási eredmény optimális abban az értelemben, hogy $a > 1$-re a $g'(0) = 0$ feltétel mellett a zéró megoldás intstabil. Továbbá, megemlítünk néhány nyitott problémát.

A 3. Fejezetben egy hálózat-modellt tekintünk, amelyet eredetileg Ranjan, La és Abed vezetett be a [31, 30] cikkekben. Ez egyetlen felhasználót és egyetlen szervert tartalmaz. A felhasználó $x(t)$ rátával küld adatokat feldolgozásra a szervernek. A szerver a bejövő adatokat c kapacitással dolgozza fel. Kelly [19] bevezette a feldolgozás $U(x)$ hasznosságát és $p(x)$ egységárát, illetve javasolt egy végfelhasználói rátaszabályzási algoritmust egy differenciálegyenlet formájában.

Amint az $x(t)$ ráta a szerver kapacitása fölé nő, a szerverhez beérkező adatok a feldolgozás előtt egy $y(t)$ hosszú sort alkotnak. Tegyük fel, hogy azon adategység, amely fel lett dolgozva, és amelyről a felhasználó a t időben egy visszajelzést, $\tau(t)$ idővel korábban ért a sorhoz, $y(t - \tau(t))$ hosszú sort talált, és $z(t) = (1/c)y(t - \tau(t))$ időt töltött sorban állással, mielőtt megkezdődött a feldolgozása. Ekkor a modell a

$$\dot{x}(t) = \kappa\big[x(t)U'(x(t)) - x(t - r_0 - z(t) - r_1)p(x(t - z(t) - r_1))\big], \tag{3.1.4}$$

$$\dot{y}(t) = \begin{cases} x(t - r_0) - c & \text{if } 0 < y(t) < q, \\ [x(t - r_0) - c]^+ & \text{if } y(t) = 0, \\ -[x(t - r_0) - c]^- & \text{if } y(t) = q, \end{cases} \tag{3.1.2}$$

$$z(t) = \frac{1}{c} y(t - z(t) - r_1) \tag{3.1.3}$$

egyenletrendszerrel írható le.

Előbb egy valamivel általánosabb, az

$$\dot{x}(t) = F(x_t, y_t) \tag{3.1.5}$$

és (3.1.2) egyenletekből álló rendszert tekintjük $X \times Y$-ban. Az $X \times Y$ fázistér az összes lehetséges (x_t, y_t) szegmenst tartalmazza.

Azért, hogy lássuk, hogy a (3.1.4), (3.1.2), (3.1.3) rendszer a (3.1.5), (3.1.2) egy speciális esete, bevezetjük a $Z = [0, q/c] \subset \mathbb{R}$ halmazt mint a $z(t)$ változó állapotterét. Egy döntő tény az, hogy a $\sigma : Y \to Z$ Lipschitz-folytonos leképezés létezik és egyértelmű a

$$\sigma(\psi) = \frac{1}{c}\psi(-\sigma(\psi) - r_1) \qquad (\psi \in Y).$$

feltétellel. Ekkor a $X \times Y$ fázistéren a (3.1.5), (3.1.2) rendszer egy $(x, y) : [-r, \infty) \to \mathbb{R}^2$ megoldására $z(t) = \sigma(y_t)$-t definiálva $t \geq 0$-ra, a (3.1.3) egyenlet mindig teljesül minden $t \geq 0$-ra.

Tegyük fel, hogy adott egy $G : X \times Z \to \mathbb{R}$ leképezés úgy, hogy a speciális

$$F : X \times Y \ni (\varphi, \psi) \mapsto G(\varphi, \sigma(\psi)) \in \mathbb{R},$$

választással a (H1)–(H4) hipotézisek teljesülnek. Ebben az esetben a (3.1.5), (3.1.2) rendszer ekvivalens a

$$\dot{x}(t) = G(x_t, z(t)), \tag{3.1.6}$$

(3.1.2) és (3.1.3) egyenletekből összeállított rendszerrel. Ekkor, az $X \times Y$ fázistérben, minden $(\varphi, \psi) \in X \times Y$-ra a (3.1.6), (3.1.2), (3.1.3) rendszernek létezik és egyértelmű az $x^{\varphi,\psi}[-r, \infty) \to \mathbb{R}$, $y^{\varphi,\psi} : [-r, \infty) \to \mathbb{R}$, $z^{\varphi,\psi} : [0, \infty) \to \mathbb{R}$ megoldása, ahol $\left(x^{\varphi,\psi}, y^{\varphi,\psi}\right)$ a (3.1.5), (3.1.2) rendszer megoldása, és $z^{\varphi,\psi}(t) = \sigma(y_t^{\varphi,\psi})$, $t \geq 0$.

A 3.3. Szakaszban megmutatjuk, hogy a (H1)–(H4) hipotézisek mellett bármely $(\varphi, \psi) \in X \times Y$-ra a (3.1.5), (3.1.2) rendszernek létezik és egyértelmű a maximális $\left(x^{\varphi,\psi}, y^{\varphi,\psi}\right) : [-r, \infty) \to \mathbb{R}^2$ megoldása, lásd a 3.3.5 Tételt. A megoldások egy folytonos

$$\Phi : [0, \infty) \times X \times Y \ni (t, \varphi, \psi) \mapsto \left(x_t^{\varphi,\psi}, y_t^{\varphi,\psi}\right) \in X \times Y,$$

félfolyamot definiálnak, és a $\Phi(t, \cdot, \cdot) : X \times Y \to X \times Y$ megoldásoperátorok Lipschitz-folytonosak minden $t \geq 0$-ra.

A 3.3.11 Tételben azt is megmutatjuk, hogy a (3.1.6), (3.1.2), (3.1.3) rendszer nem csak az $X \times Y$, hanem az $X \times Z$ fázistéren is vizsgálható a megoldás egy másik fogalmával. A legfontosabb technikai eredmény az, hogy létezik és egyértelmű az $\gamma : X \times Z \to Y$ Lipschitz-folytonos leképezés úgy, hogy $\psi = \gamma(\varphi, \zeta)$ teljesíti a $\psi(s) = c\zeta$ egyenlőséget $s \in [-r, -\zeta - r_1]$-re, és a (3.1.2) teljesül majdnem mindenhol a $[-\zeta - r_1, 0]$ intervallumon. Sőt, $\zeta = (1/c)\psi(-\zeta - r_1)$. Ez azt jelenti, hogy a sorhossz rekonstruálható a ráta múltjából (ami $\varphi \in X$) és a jelenlegi várakozási időből. A két különböző fázistér között ható h és k leképezések Lipschitz-folytonosak, h injektív, de k nem, $k \circ h = \mathrm{id}_{X \times Z}$, és $h \circ k\big|_{h(X \times Z)} = \mathrm{id}_{h(X \times Z)}$. Ekkor minden $(\varphi, \zeta) \in X \times Z$-re létezik és egyértelmű a (3.1.6), (3.1.2), (3.1.3) rendszer $x_0^{\varphi,\zeta} = \varphi$, $z^{\varphi,\zeta}(0) = \zeta$ kezdeti feltételt teljesítő megoldása. Továbbá

$$\Psi : [0, \infty) \times X \times Z \ni (t, \varphi, \zeta) \mapsto \left(x_t^{\varphi,\zeta}, z^{\varphi,\zeta}(t)\right) \in X \times Z$$

folytonos félfolyam $X \times Z$-n, és $\Psi(t, \varphi, \zeta) = k(\Phi(t, h(\varphi, \zeta)))$ minden $t \geq 0$-ra.

A 3.4. Szakaszban feltesszük, hogy $r_0 = 0$, $r_1 = 1$, és tekintjük a (3.1.4), (3.1.2), (3.1.3) rendszert. Az $r_0 = 0$ feltétel garantálja, hogy a (3.1.4) egyenletnek egyetlen késleltetése van, $r_1 = 1$ elérhető az idő újraskálázásával. Ekkor az új $v = x - x_*$ változóra átírhatjuk a rendszerünket. A 3.3.11. tételből következik, hogy a (3.1.7), (3.1.8), (3.1.9) rendszer jól definiált az $\mathcal{X} \times Z$ fázistéren.

A (3.1.7), (3.1.8), (3.1.9) rendszer egy (v, z) megoldását *lassan oszcillálónak* nevezzük, ha v bármely két t_1, t_2 zéróhelyére $t_1 < t_2$ esetén a $z(t_2) + 1 < t_2 - t_1$ egyenlőtlenség teljesül. Ez azt jelenti, hogy v egymást követő zéróhelyei közt a távolság nagyobb, mint a késleltetés.

Bevezetjük a W és $W_0 = W \cup \{(0,0)\}$ halmazokat. Ekkor minden $(\varphi, \zeta) \in W$-ra a $v = v^{\varphi,\zeta} : [-r, \infty) \to \mathbb{R}$, $z = z^{\varphi,\zeta} : [0, \infty) \to \mathbb{R}$ megoldás lassan oszcilláló végtelen sok zéróhellyel. A v második zéróhelye $(0, \infty)$-en, t_2 meghatározza $t_2^* > t_2$-t úgy, hogy $t_2 = t_2^* - z(t_2^*) - 1$, és egy $P : W_0 \to W_0$ visszatérési leképezést tudunk definiálni. P egy nemtriviális fixpontja egy lassan oszcilláló periodikus megoldásnak felel meg. Egy klasszikus eszköz, amit itt alkalmazunk, Browder nem-taszító fixpont-tétele. A 3.4. Szakasz egy nagy része $\mathcal{X} \times Z$ egy megfelelő részhalmazának konstrukciójáról szól, ahol Browder tétele alkalmazható.

Az egy döntő eredmény, hogy $P(\varphi, \zeta)$ nem csökkenhet túl gyorsan: vannak olyan $\theta > 0$, $\rho > 0$ konstansok, hogy $v^{\varphi,\zeta}(t_2^*) \geq \theta\,(\varphi(0))^\rho$ minden $(\varphi, \zeta) \in W$-re. Ez a tény lehetővé teszi, hogy konstruáljunk egy megfelelő C^2-sima α függvényt. A $\mathcal{X} \times Z$ kompakt W_{α,K_1} és W_{α,K_0} részhalmazait definiálva, teljesül a $P(W_{\alpha,K_1}) \subseteq W_{\alpha,K_0}$ tartalmazás. Azonban W_{α,K_1} és W_{α,K_0} nem konvex. [25]-t követve, a $C_{[-1,0]} \times \mathbb{R}$ tér V_{α,K_1} részhalmaza kompakt és konvex. A V_{α,K_1} halmaz W_{α,K_1}-be képezhető a Q nyújtó leképezéssel, amit a $Q(\psi, \zeta) = (\varphi, \zeta)$, $\varphi(s) = \psi(s/(\zeta+1))$, $s \in [-\zeta-1, 0]$, és $\varphi\big|_{[-r,-\zeta-1]} \equiv 0$ képletek adnak meg. Az R zsugorító leképezés, melyet a $R(\varphi, \zeta) = (\psi, \zeta)$, $\psi(s) = \varphi((\zeta+1)s)$, $s \in [-1, 0]$ képletek adnak, W_{α,K_0}-t V_{α,K_1}-ba képezi. Browder tétele alkalmazható a $\Pi = R \circ P \circ Q$ leképezés nem-taszító fixpontjának megtalálására V_{α,K_1}-ben. Ez egyben a P egy nem-taszító fixpontját is adja W_{α,K_1}. A nem-taszító fixpont nemtriviális, feltéve, hogy $(0,0) \in W_{\alpha,K_1}$ taszító. A $(0,0) \in W_{\alpha,K_1}$ taszító tulajdonsága szokásos módon jön az $\dot{v}(t) = -f(v(t)) - g(v(t-1))$ egyenlet zéró megoldásának taszító tulajdonságából. Így kimondhatjuk a fő eredményünket a 3.4.17 Tételben.

Végül, a 3.5. Szakasz példákat ad.

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