Clones of small posets

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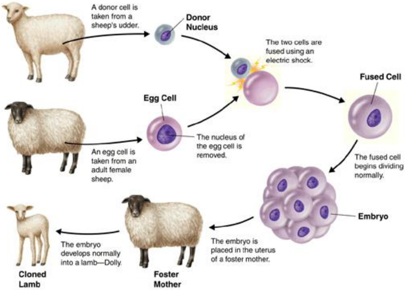
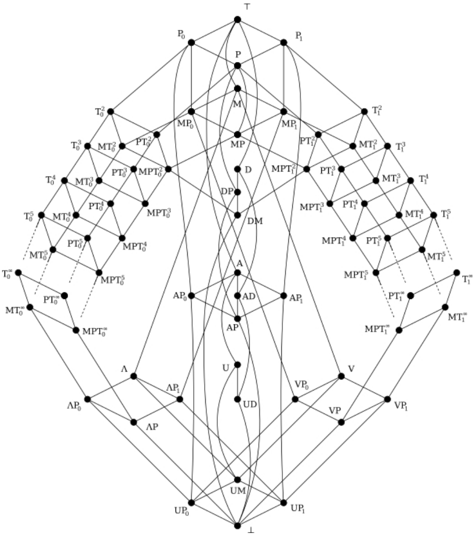
Clones that normal people have in mind



Clones that we choose to spend time with



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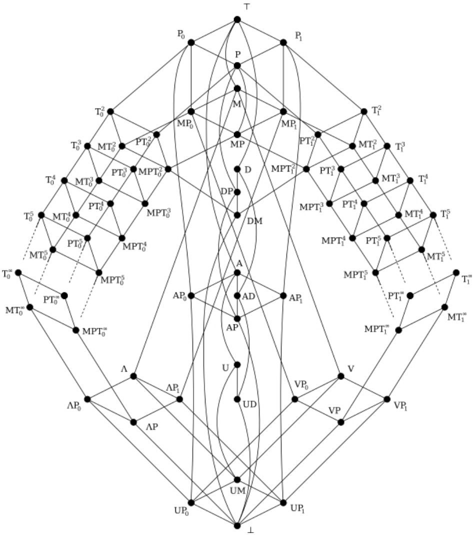
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AP_0
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Clones on a three element set

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The first problematic bounded poset has turned out to be:

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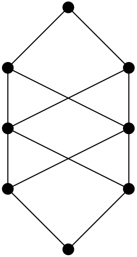

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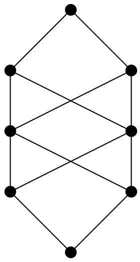

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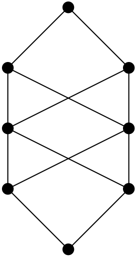

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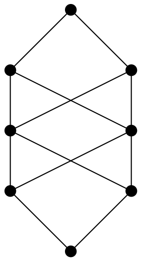

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In 1993, L. Zádori gave a full description of series-paralell posets that have non-finitely generated clones.
No one has found nonfinitely generated maximal clones since, though one may conjecture that there are a lot of them.

Main objective, difficulties and our approach

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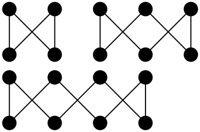

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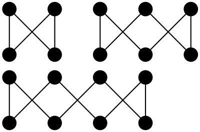

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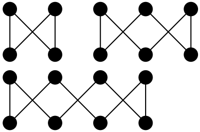

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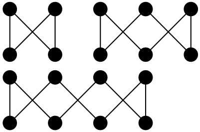

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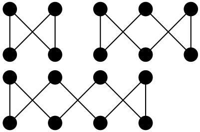

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We described the critical relations of the crowns.

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Let P be an arbitrary finite poset and $\alpha \subseteq P^n$. Then α is invariant if and only if there exists a finite poset Q and $(x_1, \ldots, x_n) \in Q^n$ for which

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How does the proposition work here?

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We see that there is a connection between critical relations and obstacles: to every critical relation we can assign an obstacle.

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A partial function $P^n \to P$ is extendible monotonically if and only if it preserves all critical relations.

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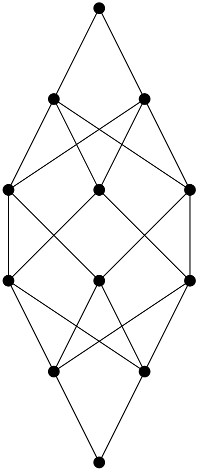
Theorem

The critical relations of C_{2k} are:

- *the unary $\emptyset$ relation,*
- *the binary critical relations: we can get them with our proposition s. t. Q is a "small" fence and $(x, y) \in Q$ is the pair of the endpoints*
- *the n-ary critical relations, $n \geq 2$: for all large range tuples $\overline{a} \in C_{2k}^n$, the relations $R_{\overline{a}}$.*

Locked 6-crown: $1 + 2 + C_6 + 2 + 1$



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We felt this might be possible...

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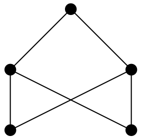
The clone and also the idempotent clone of the house poset



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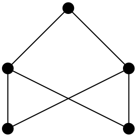

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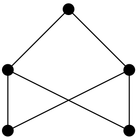

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So our observation does not extend to half-bounded posets.

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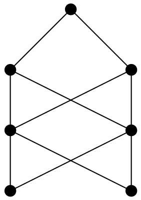

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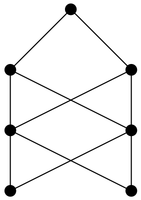

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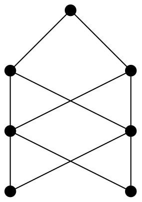
We also know that its idempotent clone and reduced idempotent clone coincide.

Question

We know from Zádori's paper that the clone of



is non-finitely generated.
We also know that its idempotent clone and reduced idempotent clone coincide. QUESTION. Is this clone finitely generated?

Thank you!